BioE 1410 - Homework 4 - Answers

1.

$$a_1 = -\frac{j}{2} \quad a_{-1} = \frac{j}{2} \quad a_2 = \frac{1}{2} \quad a_{-2} = \frac{1}{2}$$

$$b_1 = \frac{1}{2} \quad b_{-1} = \frac{1}{2} \quad b_2 = -\frac{j}{2} \quad b_{-2} = \frac{j}{2}$$

$$\boxed{\overline{P}(x(t)) = \frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2}$$ Parseval's Relation

$$\overline{P}(x(t)) = \overline{P}(y(t)) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

2.

A.

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{Fs} jk\omega_0 a_k}$$

$$d_1 = -\frac{j}{2}(jk\omega_0) = \frac{\omega_0}{2}$$

$$d_{-1} = \frac{\omega_0}{2}$$

$$d_2 = 2j\frac{\omega_0}{2}$$

$$d_{-2} = -2j\frac{\omega_0}{2}$$

$$z(t) = \omega_0 \cos(\omega_0 t) - 2\omega_0 \sin(2\omega_0 t)$$

B.

$$\boxed{Ax(t) + Bx(t) \xleftrightarrow{Fs} Aa_k + Bb_k}$$

$$d_1 = -\left(\frac{1+2j}{2}\right)(jk\omega_0) = \left(\frac{j-2}{2}\right)\omega_0$$

$$d_{-1} = \left(\frac{-j-2}{2}\right)\omega_0$$

$$d_2 = \left(\frac{2-4j}{2}\right)\omega_0$$

$$d_{-2} = \left(\frac{2+4j}{2}\right)\omega_0$$

$$z(t) = -\omega_0 2\cos(\omega_0 t) - \omega_0 \sin(\omega_0 t) + \omega_0 4\sin(2\omega_0 t) + 2\omega_0 \cos(2\omega_0 t)$$

C.

$$\boxed{\int x(t)dt \xleftrightarrow{Fs} \frac{1}{jk\omega_0} a_k}$$

$$d_1 = \frac{1}{jk\omega_0}\frac{1}{2} = -\frac{j}{2\omega_0}$$

$$d_{-1} = \frac{1}{jk\omega_0}\frac{1}{2} = \frac{j}{2\omega_0}$$

$$d_2 = \frac{1}{jk\omega_0}\left(-\frac{j}{2}\right) = -\frac{1}{4\omega_0}$$

$$d_{-2} = \frac{1}{jk\omega_0}\left(-\frac{j}{2}\right) = -\frac{1}{4\omega_0}$$

$$z(t) = \frac{1}{\omega_0}\sin(\omega_0 t) - \frac{1}{2\omega_0}\cos(2\omega_0 t)$$

3. $\boxed{x(t-\tau) \xleftrightarrow{Fs} e^{-jk\omega_0\tau} a_k}$

$$T_0 = 2$$

$$\omega_0 = \frac{2\pi}{T_0} = \pi$$

$$x(t) = 17 + 6\cos(\pi t) - 4\sin(2\pi t)$$

delay $\tau = 1 = \frac{T_0}{2}$

$$b_0 = e^0 a_0 = 17$$

$$b_1 = e^{-j\pi} a_1 = -3$$

$$b_{-1} = e^{j\pi} a_{-1} = -3$$

$$b_2 = e^{-j2\pi} a_2 = 2j$$

$$b_{-2} = e^{j2\pi} a_{-2} = -2j$$

$$y(t) = 17 - 6\cos(\pi t) - 4\sin(2\pi t)$$

4. **A.**

$$a_1 = -\frac{j}{2} \quad a_{-1} = \frac{j}{2} \quad \omega_0 = 4\pi \quad T = \frac{2\pi}{4\pi} = \frac{1}{2}$$

B.

$$a_0 = 4 \quad a_1 = -\frac{1}{2} \quad a_{-1} = -\frac{1}{2} \quad a_3 = -\frac{3j}{2} \quad a_{-3} = \frac{3j}{2}$$

$$\omega_0 = 2\pi \quad T = 1$$

C. $\boxed{x(t - t_0) \xleftrightarrow{Fs} e^{-jk\omega_0 t_0} a_k}$

Careful; time-shifting from $x(t)$ to $x(t-t_0)$ is the proper approach, but $\cos(4\pi t - \pi)$ must first be converted to $\cos\left(4\pi\left[t - \frac{1}{4}\right]\right)$ to get a proper value for t_0 of ¼.

$$a_1 = \frac{1}{2}\mathrm{e}^{-\mathrm{jk}\omega_0 t_0} = \frac{1}{2}\mathrm{e}^{-\mathrm{j}\pi} = -\frac{1}{2} \qquad a_{-1} = \frac{1}{2}\mathrm{e}^{-\mathrm{jk}\omega_0 t_0} = \frac{1}{2}\mathrm{e}^{\mathrm{j}\pi} = -\frac{1}{2}$$

$$t_0 = \frac{1}{4} \qquad \omega_0 = 4\pi \qquad T = \frac{1}{2}$$

5. **A.** $\left|-\frac{j}{2}\right|^2 + \left|\frac{j}{2}\right|^2 = \frac{1}{2}$

B. $16 + \frac{1}{2} + \frac{9}{2} = 21$

C. $2\left|\frac{1}{2}\right|^2 = \frac{1}{2}$

6. **A.** $b_1 = -j \quad b_{-1} = j$

B. $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

C. 12π

D. twice the amplitude from problem 5 (for function 4A) means 4 times the power

$$\frac{1}{2} \cdot 4 = 2$$

For questions 7, use the following notation for the coefficients of the Fourier series.

$$\boxed{x(t) \xleftrightarrow{FS} a_k}$$

7. Find all non-zero a_k and for

$$x(t) = 2 - \sin(\omega_0 t) + \cos(2\omega_0 t) - 2\sin(2\omega_0 t)$$

Then, using Parseval's relation, find the average power of $x(t)$.
Sketch the Fourier Transform for $x(t)$ labeling all axes and coordinates(Real + Imag. parts)
Is the total energy of $x(t)$ finite or infinite?

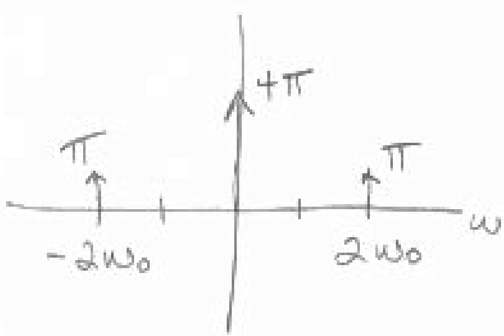
$a_0 = 2$ $\qquad |a_0| = 2$

$a_1 = +\frac{j}{2} \quad a_{-1} = -\frac{j}{2} \qquad |a_1| = \frac{1}{2} = |a_{-1}|$

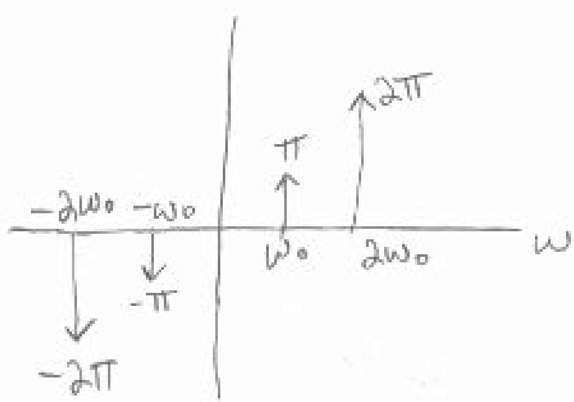
$a_2 = \frac{1}{2} + j \quad a_{-2} = \frac{1}{2} - j \qquad |a_2| = \sqrt{1\frac{1}{4}} = |a_{-2}|$

$$P = \sum_{k=-\infty}^{+\infty} |a_k|^2 = 2^2 + 2\left(\frac{1}{2}\right)^2 + 2\left(\sqrt{1\frac{1}{4}}\right)^2 = 7$$

$Re\{X(\omega)\}$



$Im\{X(\omega)\}$



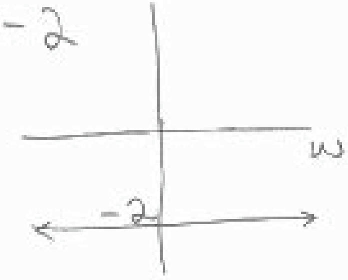
energy is infinite

8. Given that $x(t)=\sin(2\pi t)$ and $h(t)=-2\delta(t)$, write equations for the Fourier transforms $X(\omega)$ and $H(\omega)$ and graph both spectra labeling all axes and coordinates. Write an equation and graph both the output of the system $y(t)=x(t)*h(t)$ and its Fourier transform $Y(\omega)$.

$$X(\omega) = -\pi j\,\delta(2\pi-\omega) + \pi j\,\delta(2\pi+\omega)$$

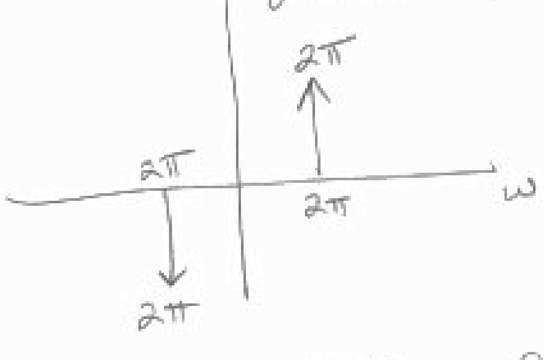
$\mathrm{Im}\{X(\omega)\}$

$$H(\omega) = -2$$

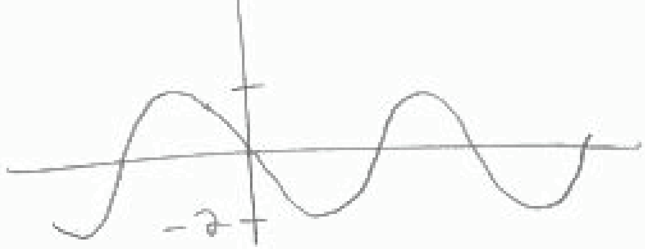


$$Y(\omega) = X(\omega)H(\omega) = 2\pi j\,\delta(2\pi-\omega) - 2\pi j\,\delta(2\pi+\omega)$$

$\mathrm{Im}\{Y(\omega)\}$



$$y(t) = -2\sin(2\pi t)$$



(9) $$|X(\omega)| = \left[\left(\mathrm{Re}\{X(\omega)\} \right)^2 + \left(\mathrm{Im}\{X(\omega)\} \right)^2 \right]^{\frac{1}{2}}$$

since $X(-\omega) = X(\omega)^*$ for real $x(t)$

$$|X(-\omega)| = \left[\left(\mathrm{Re}\{X(\omega)\} \right)^2 + \left(\mathrm{Im}\{-X(\omega)\} \right)^2 \right]^{\frac{1}{2}}$$

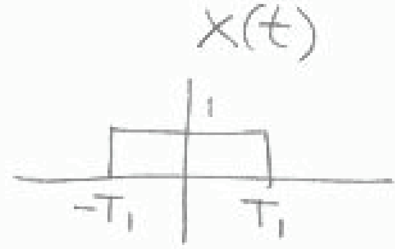
$$= X(\omega) \Rightarrow \text{even}$$

(10) as in Problem (9)

$$\measuredangle X(\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\{X(\omega)\}}{\mathrm{Re}\{X(\omega)\}} \right)$$

$$\measuredangle X(-\omega) = \tan^{-1}\left(\frac{-\mathrm{Im}\{X(\omega)\}}{\mathrm{Re}\{X(\omega)\}} \right) = -\measuredangle X(-\omega)$$

(11) $$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$



$$= \int_{-T_1}^{T_1} e^{-j\omega t} dt$$

$$= -\frac{1}{j\omega} e^{-j\omega t} \Big|_{-T_1}^{T_1}$$

$$= \frac{1}{j\omega}\left[e^{j\omega T_1} - e^{-j\omega T_1} \right]$$

$$= \frac{2}{\omega}\left[\frac{e^{j\omega T_1} - e^{-j\omega T_1}}{2j} \right] = \frac{2\sin \omega T_1}{\omega}$$

expressed as a sinc function

$$\operatorname{sinc} \lambda \triangleq \frac{\sin \pi \lambda}{\lambda \pi}$$

$$\lambda = \frac{\omega T_1}{\pi}$$

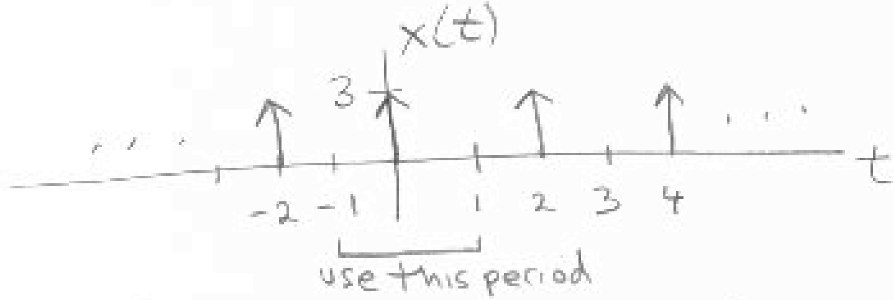
$$\frac{2 \sin \omega T_1}{\omega} = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi} \right)$$

(12)

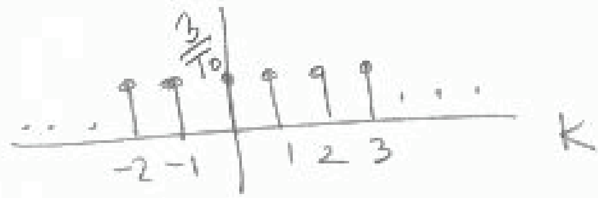
$$x(t) = 3\sum_{n=-\infty}^{+\infty} \delta(t-2n)$$

$$T_0 = 2$$

$$\omega_0 = \frac{2\pi}{T_0} = \pi \ \frac{\text{Radians}}{\text{sec}}$$



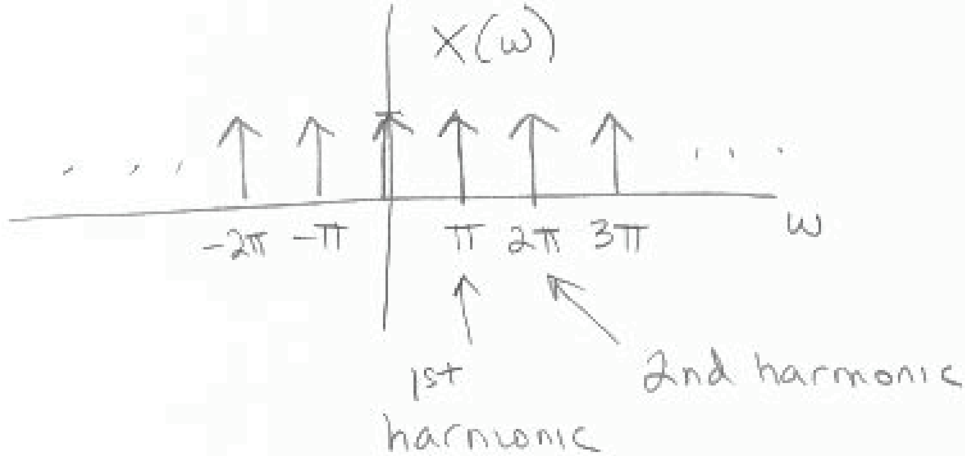
$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}\int_{T_0} 3\delta(t)e^{-jk\omega_0 t}\,dt = \frac{3}{T_0}$$



average value of $x(t)$

$$X(\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

$$= \frac{6\pi}{T_0}\sum \delta(\omega - k\pi)$$



(13) $x(t) = \begin{cases} 1, & |t| < \frac{T_0}{4} \\ 0, & \frac{T_0}{4} \le |t| \le \frac{T_0}{2} \\ x(t+T_0), & \text{elsewhere} \end{cases} \quad \xleftrightarrow{FS} \quad \boxed{a_k = \frac{1}{2}\operatorname{sinc}\left(\frac{k}{2}\right)}$

$\omega_0 = \frac{2\pi}{T_0}$

$T_0 \omega_0 = 2\pi$

PROOF $a_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}\, dt$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{4}}^{+\frac{T_0}{4}} e^{-jk\omega_0 t}\, dt = -\frac{1}{jk\omega_0 T_0} e^{-jk\omega_0 t}\Big|_{-\frac{T_0}{4}}^{+\frac{T_0}{4}}$$

$$= \frac{1}{jk2\pi}\left[e^{jk\omega_0 \frac{T_0}{4}} - e^{-jk\omega_0 \frac{T_0}{4}} \right]$$

$$= \frac{1}{k\pi}\left[\frac{e^{\frac{jk\pi}{2}} - e^{-\frac{jk\pi}{2}}}{2j} \right] = \frac{1}{k\pi}\sin\left(\frac{k\pi}{2}\right)$$

let $\lambda = \frac{k}{2}$ where $\operatorname{sinc}\lambda \triangleq \frac{\sin \pi\lambda}{\pi\lambda}$

$$a_k = \frac{1}{2}\left[\frac{\sin\pi\lambda}{\pi\lambda}\right] = \frac{1}{2}\operatorname{sinc}(\lambda) =$$

$$= \frac{1}{2}\operatorname{sinc}\left(\frac{k}{2}\right)$$

(14) $H_1(\omega) = j\omega$

$H_2(\omega) = \frac{1}{j\omega}$ } inverses of each other

$$H(\omega) = H_1(\omega) H_2(\omega) = 1$$

$$h(t) = \delta(t)$$

A subtlety of this problem is that integration of a signal with a DC component $X(\omega) \neq 0$ produces ∞. (see Schaum, eq. 5.57). Luckily the output of $h_1(t)$, the differentiator, cannot have a DC component unless its input were unbounded with an ever-increasing (or decreasing) value.

(15) $H(\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt$

$$= \int_{-\infty}^{+\infty} e^{-at} u(t) e^{-j\omega t} dt = \int_{0}^{+\infty} e^{-(a+j\omega)t} dt$$

$$= \frac{1}{a+j\omega}$$ favors low frequencies

(16) $X(\omega) = \pi(1-2j)\delta(\omega-3) + \pi(1+2j)\delta(\omega+3)$

$Y(\omega) = X(\omega)^* = \pi(1+2j)\delta(\omega-3) + \pi(1-2j)\delta(\omega+3)$

The sin gets flipped but the cos doesn't.