1. The Phasor – What is a complex exponential?

We need calculus. Calculus is the engine behind Newton's laws, Maxwell's equations, chemical rates of reactions, and many other essential relationships in the universe. Calculus has two fundamental operations, *differentiation* and *integration*, which are really opposite expressions of the same thing. Consider the functions $f(t)$ and $g(t)$ such that

$$f(t) = \frac{dg(t)}{dt}. \tag{1-1}$$

Differentiation is the opposite of integration, put mathematically as

$$g(t) = \int_{-\infty}^{t} f(\tau)d\tau. \tag{1-2}$$

As shown in Figure 1, $g(t)$ is the area under $f(\tau)$ between $\tau = -\infty$ and $\tau = t$. As t is moved to the right (the arrow in Figure 1), the rate at which that area $g(t)$ changes is simply $f(t)$ itself. The derivative of the integral is itself.

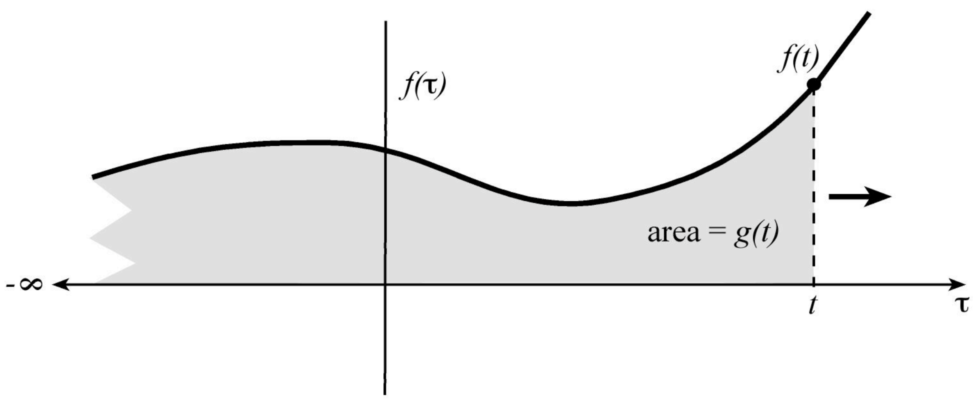


Figure 1-1. The derivative of the area $g(t)$ as t moves to the right is $f(t)$.

To solve real-world relationships in calculus, which are typically posed as *differential equations*, we need *exponentials*. Exponentials are nice because they reduce multiplication to addition. We learn in grade school that raising x to an integer power permits us to add exponents,

$$x^2 \cdot x^3 = x^5 \tag{1-3}$$

Really, we are just keeping track of how many times x has been multiplied by itself:

$$(x \cdot x)\cdot(x \cdot x \cdot x) = (x \cdot x \cdot x \cdot x \cdot x), \quad 2+3=5. \tag{1-4}$$

But what does it mean to raise x to the 1.5^{th} power? This is not quite as easy to grasp. How do we multiply x by itself 1.5 times? Here we are taught that the *exponential function* x^t connects the integer powers of x into a smooth curve, for any real number t. So for example, when $x = 2$ the integer powers of 2 are connected by the continuous function 2^t as shown in Figure 2.

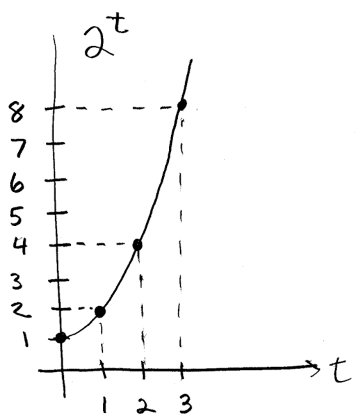


Figure 1-2 Continuous function 2^t exponential connects the integer powers of 2^t

The exponential x^t is still endowed with the ability to add exponents for a given base x, even when t is not an integer,

$$x^{1.5} \cdot x^{3.2} = x^{4.7}. \tag{1-5}$$

though we may have lost the childhood clarity of simple counting and we take it on faith. The word *calculus* means *stone* in Latin, and it would be nice if calculus were really as simply as counting stones. We will do the best we can.

What about the derivative of an exponential? When taking the derivative of x^2 with respect to x we are taught that

$$\frac{d(x^2)}{dx} = 2x\ . \tag{1-6}$$

This is shown, quite believably, by recalling that dx is the limit of Δx as $\Delta x \to 0$. Thus,

$$\frac{d(x^2)}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = 2x\ . \tag{1-7}$$

We generalize this to any integer power of x in the basic rule

$$\frac{d(x^n)}{dx} = nx^{n-1} \tag{1-8}$$

of which Equation (1-6) is just a special case.

What happens when the variable is not the base, as in x^2, but rather when it is the exponent, as in 2^t? What is its derivative $d2^t/dt$? If we examine 2^t as t assumes the values $0, 1, 2, 3, \ldots$, and compute for the change $\Delta(2^t)$ at each value of t

$$\begin{array}{llllllllll}
t & 0 & & 1 & & 2 & & 3 & & 4.... \\
2^t & 1 & & 2 & & 4 & & 8 & & 16.... \\
 & & \backslash\ / & & \backslash\ / & & \backslash\ / & & \backslash\ / & \\
\Delta(2^t)/\Delta t & & 1 & & 2 & & 4 & & 8...... &
\end{array}$$

Figure 1-2 The change in the exponential 2^t is itself and exponential.

we find that $\Delta(2^t)/\Delta t$ is also an exponential. Indeed, it the same exponential but somewhat shifted to the right. In a similar fashion, it turns out that the derivative of any exponential is also an exponential.

But to solve differential equations, we need to find a particular value for the base, the number we call "e", to serve as the base for an exponential that will satisfy the equation

$$\frac{d(e^t)}{dx} = e^t \tag{1-9}$$

The function e^t is the only function whose derivative is itself. We saw that the derivative of 2^t lagged a little behind it (to the right). Intuitively, e will have to be a little bigger than 2, to make it go up faster.

How do we find the actual numerical value of e? We can express e^t without having t in the exponent but only in the base, in which case we know how to find the derivative, as already described. Let us assume there is some polynomial expression for e^t:

$$e^t = a_0 t^0 + a_1 t^1 + a_2 t^2 + a_3 t^3 .... = \sum_{n=0}^{\infty} a_n t^n . \tag{1-10}$$

Simply by knowing the derivative of e^t is itself, we can solve for the values of all the a_i's, in the following manner.

Setting $t = 0$ leaves only the zero-order term,

$$e^0 = a_0 t^0 , \tag{1-11}$$

and so it must be that $a_0 = 1$. When we take the derivative, the first order term $a_1 t^1$ must become the zero order term, so it must be that $a_1 = 1$ as well. Similarly, when we take the derivative, the second order term a_2t^2 becomes the first order term, and so on. The solution for the entire polynomial ends up looking like

$$e^t = 1t^0 + 1t^1 + \frac{1}{2}t^2 + \frac{1}{2\cdot 3}t^3 + \frac{1}{2\cdot 3\cdot 4}t^4 .... = \sum_{n=0}^{\infty}\frac{1}{n!}t^n \qquad \textbf{(1-12)}$$

because each term must becomes the term to the left when the derivative is taken. (The zero order term a_0x^0 is a constant, so its derivative is zero. What is the value of e? Setting $x = 1$ yields,

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} .... \qquad \textbf{(1-13)}$$

which appears to be somewhat more than 2½, and indeed is 2.71828182846...., the irrational, yet highly esteemed, constant at the heart of differential calculus.

This can be illustrated by showing that the slope of e^t is equal to itself (Figure 1-3A), or that the area under e^t is equal to itself (see Figure 1-3B) if you count the squares I think you will agree.

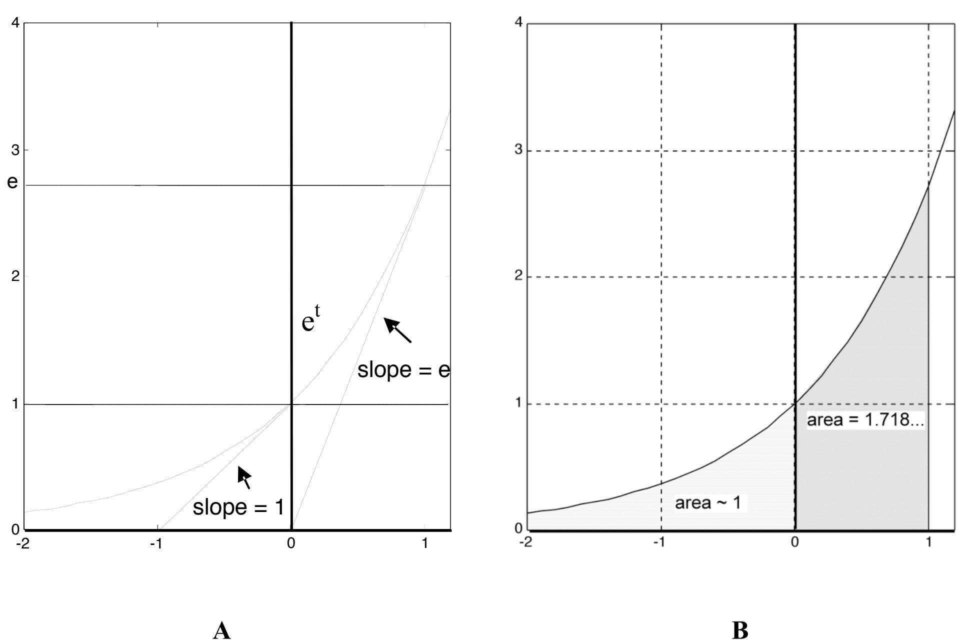
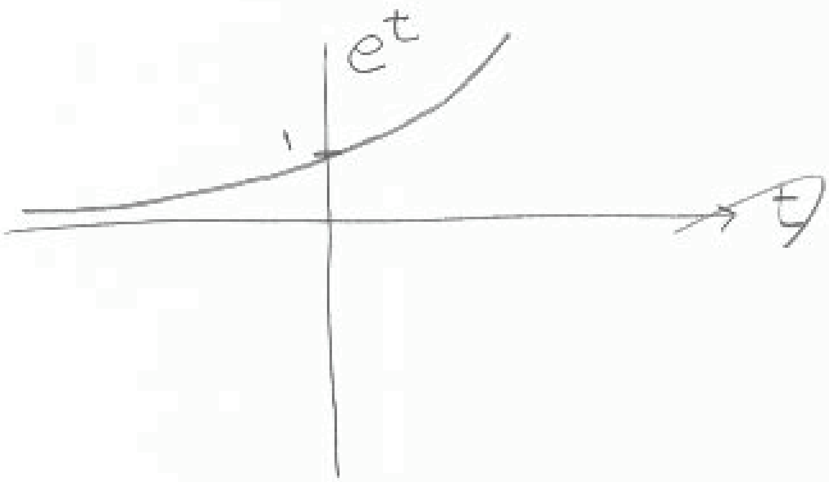
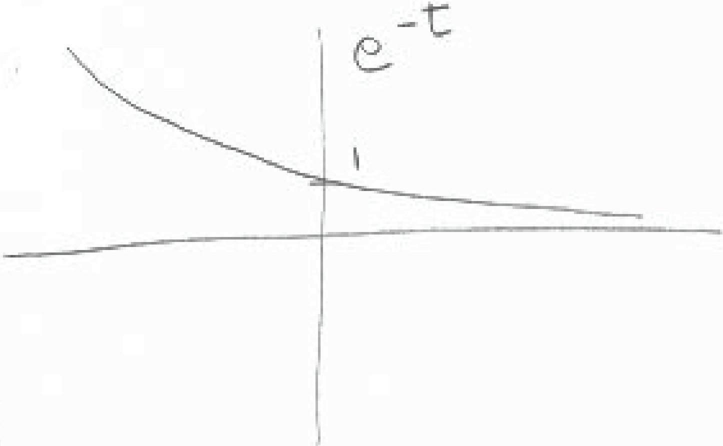


Figure 1-3 A. The slope of e^t is everywhere equal to itself.
B. The area under e^t from $-\infty$ to any value of t is e^t.





assume $a > 1$

$$e^a \times e^b = e^{(a+b)}$$

$$\frac{de^{at}}{dt} = a e^{at}$$

$$\int e^{at} dt = \frac{1}{a} e^{at}$$

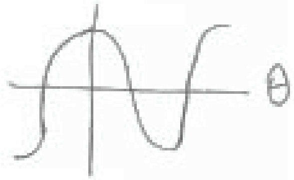
differential calculus become algebra

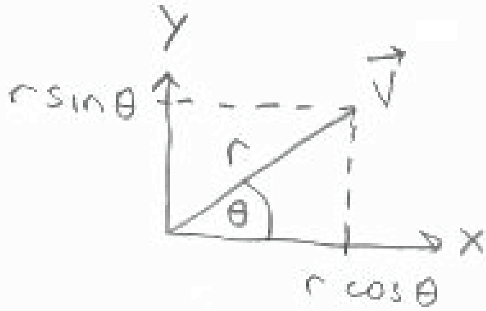
Scaling by 'a' on the 't' axis.
"speeding up" increases "slopiness", decreases area

as if you used photoshop to change the shape of the graph

Review of sin & cos

$\cos\theta = \frac{x}{r}$





$\sin\theta = \frac{y}{r}$

saying $\cos^2\theta + \sin^2\theta = 1$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

$x^2 + y^2 = r^2$ is just the pythagorean theorem

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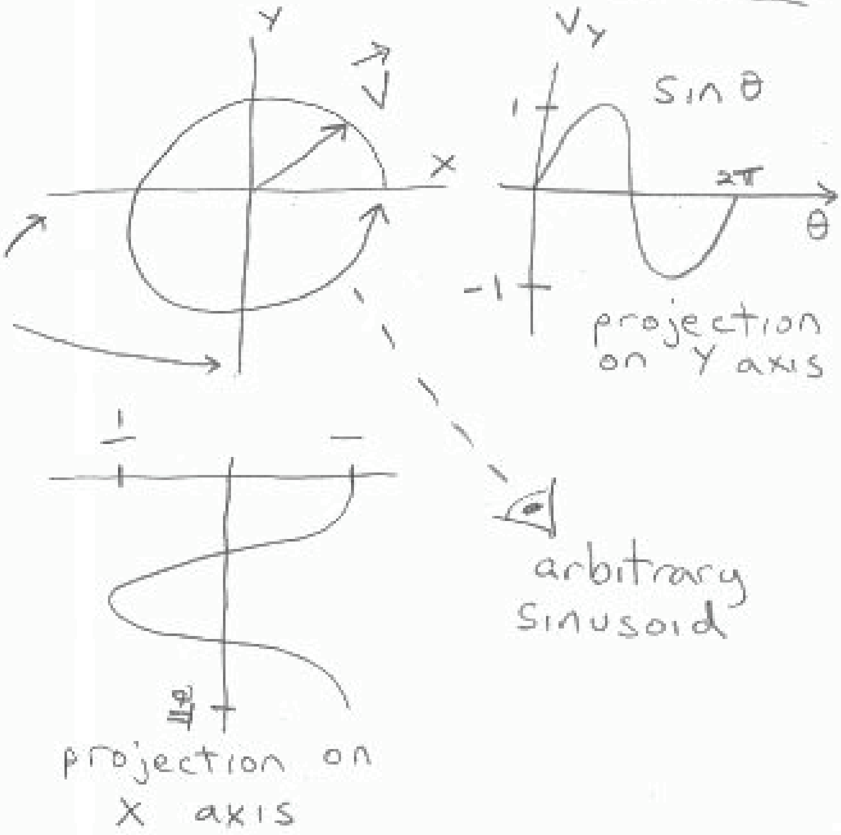
when $r = 1$

$V_x = \cos\theta$

$V_y = \sin\theta$



cardinal axes are just an arbitrary choice

Sin vs cos vs any sinusoid is just a matter of where you say $\theta = 0$

2π should be a single number

We could make up a new symbol $\nabla = 2\pi$

circumference

$2\pi r$

then we would have

circumference $= \nabla r$

$$\text{area} = \int_0^r \nabla r\,dr = \underbrace{\tfrac{1}{2}\nabla R^2}$$

θ is tradionally in Radians

2π Radians $= 360° = 1$ cycle

ω is traditionally in Radians/sec

mech. eng. uses "Ω"!! bad!!

save Ω for resistance.

$$\omega \frac{\text{radians}}{\text{sec}} = 2\pi f \frac{\text{cycles}}{\text{sec}}$$

both ω and f are "frequency"

θ is "angle" or "phase"

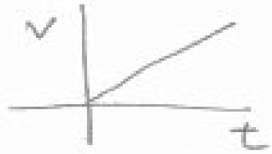
f is in cycles/second

ω is in radians/second

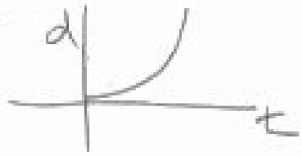
this is the familiar form for integrating a constant.

eg. Being dropped in a constant gravitational field

$v = at$



$d = \frac{1}{2}at^2$



However, we all memorize, instead

circumference $= 2\pi R$

area $= \pi R^2$

magic of sinusoids

"Superposition"

$$A \sin(\omega t) + B \cos(\omega t)$$

is a sinusoid
of freq. ω

$$\sum_i \underbrace{\overset{\text{amplitude}}{a_i} \cos(\overset{\text{frequency}}{\omega} t + \overset{\text{phase}}{\theta_i})}_{\text{any sinusoid of frequency } \omega} = \text{sinusoid of freq. } \omega.$$

time + space

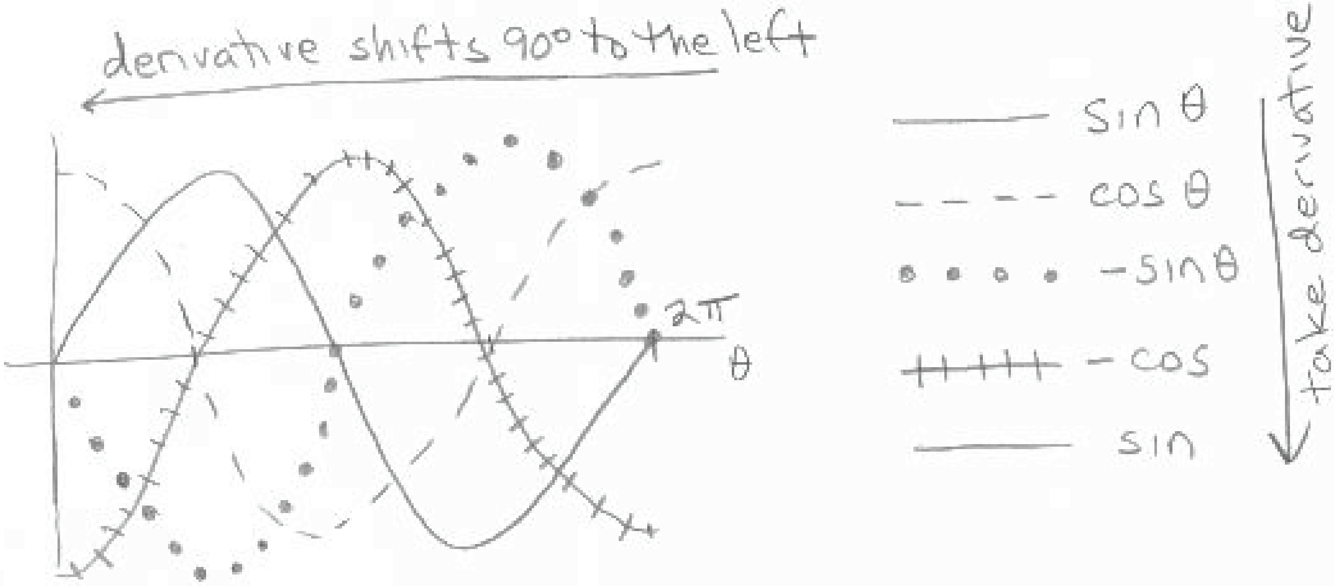
constructive / destructive
interference with
coherent radiation (sound, light)

null spots for music or radio
multi-path echos
lenses, refraction
steering ultrasound

Magic of sinusoids

"QUADRATURE"

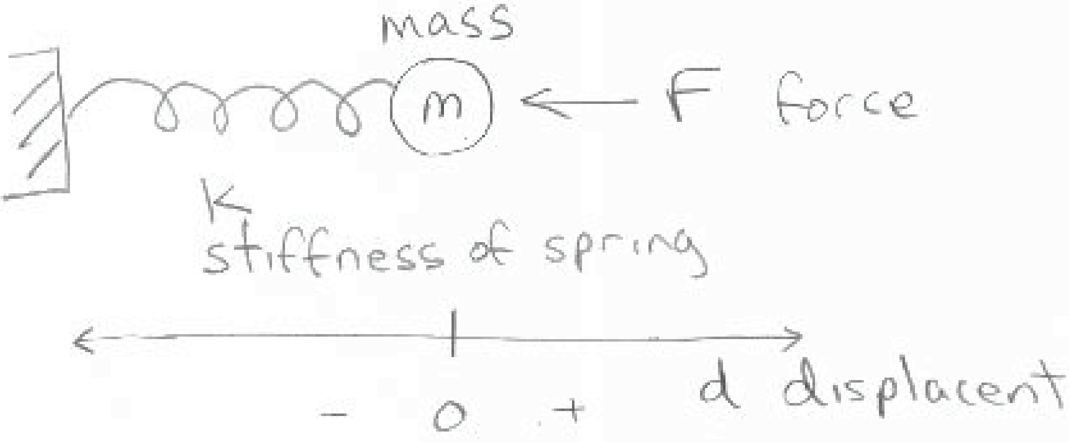
derivative shifts 90° to the left



$$90^\circ = \frac{2\pi}{4} = \frac{\pi}{2}$$

$\frac{1}{4}$ way around

Hooke Law



$$\begin{aligned} F &= ma \\ F &= -kd \end{aligned} \Rightarrow d = -\left(\frac{m}{k}\right) a$$

sinusoids result when a function is proportional to its own negative second derivative.

Pervasive in nature.