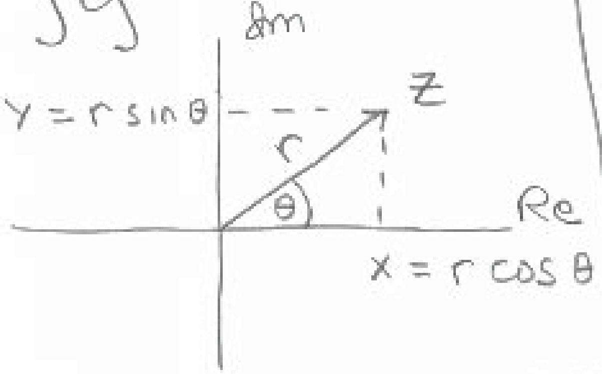
Complex numbers are represented on the cartesian complex plane like a vector, but they are <u>not</u> vectors. For example, vectors cannot simply be multiplied by each other, whereas complex numbers can.

<u>Cartesian vs. Polar coordinates of a complex number</u>

$$z = x + jy$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$



we say that $-\pi < \theta \leq +\pi$ to make it unique, though it θ actually periodic

$$\theta + k2\pi$$

$$k = 0, \pm 1, \pm 2, \ldots$$

"modulus" of z, $\rightarrow |z| = \sqrt{x^2 + y^2}$

"absolute value" is just a special case where $y = 0$.

Note: this is not $\sqrt{z^2}$, but rather the length of the line, r

$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= r\underbrace{\sqrt{\cos^2\theta + \sin^2\theta}}_{1}$$

$= r$, which is always ≥ 0

Recall "<u>superposition</u>"

sinusoids of frequency ω with differing amplitudes a and phase θ when added together always produce a sinusoid of frequency ω

$$\sum_i a_i \cos(\omega t + \theta_i) = \text{sinusoid of freq. } \omega$$

and "<u>quadrature</u>"

taking the derivative of a sinusoid shifts it $90°$ or $\frac{\pi}{2}$ to the left.

We will represent sinusoids by complex exponentials and show that the exhibit both of these qualities.

Just like with e^t, we can find polynomials for $\sin t$ and $\cos t$ by knowing that $\sin(0) = 0$, $\cos(0) = 1$, and both are equal to their negative 2nd derivatives

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} \cdots$$

$$\sin(t) = 0 + t + 0 - \frac{t^3}{3!} + 0 + \frac{t^5}{5!} \cdots$$

$$\cos(t) = 1 + 0 - \frac{t^2}{2!} + 0 + \frac{t^4}{4!} + 0 \cdots$$

$$j\sin(t) = 0 + jt + 0 - j\frac{t^3}{3!} + 0 + j\frac{t^5}{5!} \cdots$$

$$e^{jt} = 1 + jt - \frac{t^2}{2} - j\frac{t^3}{3!} + \frac{t^4}{4!} + j\frac{t^5}{5!} \cdots$$

$$\boxed{e^{jt} = \cos(t) + j\sin(t)}$$

EULER'S IDENTITY

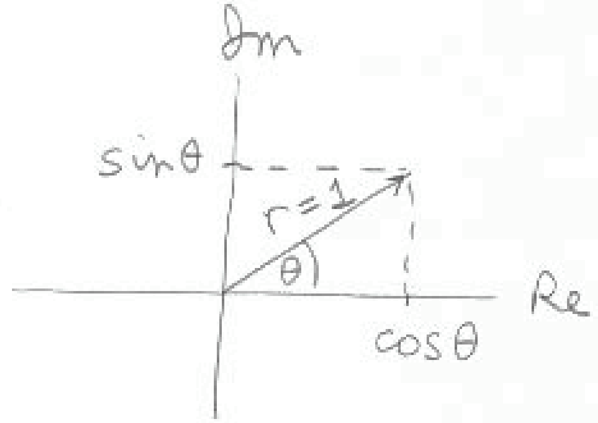
engineers use $j = \sqrt{-1}$, mathematicians use $i = \sqrt{-1}$

what does it mean to raise something to an imaginary power?

Who knows!

but <u>anything</u> algebra can do to a real number, it can also do to an imaginary number.

First,
Fixed <u>unit</u> phasor, $r = 1$

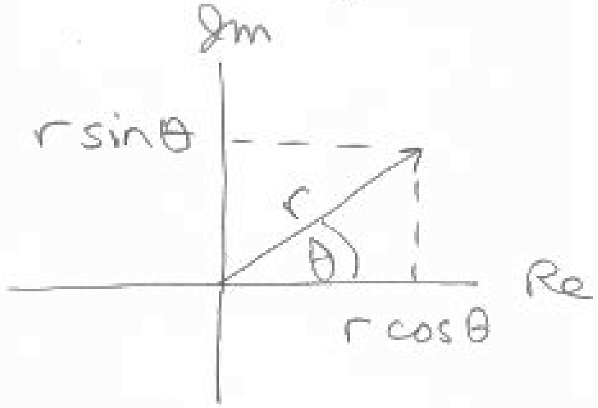


$$e^{j\theta} = \cos\theta + j\sin\theta$$

Euler's Identity

$$r = |e^{j\theta}| = 1 \quad \text{because} \quad \sin^2\theta + \cos^2\theta = 1$$
$$= \sqrt{(r\sin\theta)^2 + (r\cos\theta)^2} = r\sqrt{\sin^2\theta + \cos^2\theta} = r$$

Now, for any complex number $z = x + jy$



multiply Euler's Identity by r

$$\underbrace{r e^{j\theta}}_{\text{polar: } r,\,\theta} = \underbrace{\underbrace{r\cos\theta}_{x} + j\,\underbrace{r\sin\theta}_{y}}_{\text{cartesian: } x,\,y}$$

either way, it's just a complex number

$re^{j\theta}$ = "phasor" = "complex exponential"

why "complex"?

It looks purely imaginary.

But it can be rewritten

$$re^{j\theta} = e^{\sigma}e^{j\theta}$$

where $r = e^{\sigma}$, a positive real number.

The exponent of

$$e^{\sigma}e^{j\theta} = e^{(\sigma + j\theta)}$$

is actually any complex number

Let's review the dimensionality of phase and frequency...

θ = phase = angle

usually in radians, but can be degrees, or cycles

1 cycle = 360 degrees = 2π radians

$e^{j(\;)}$ this is phase

phase = frequency × time, as in $e^{j(\omega t)}$

$$\omega = 2\pi f$$

ω: frequency in radians/sec

f: frequency in cycles/sec

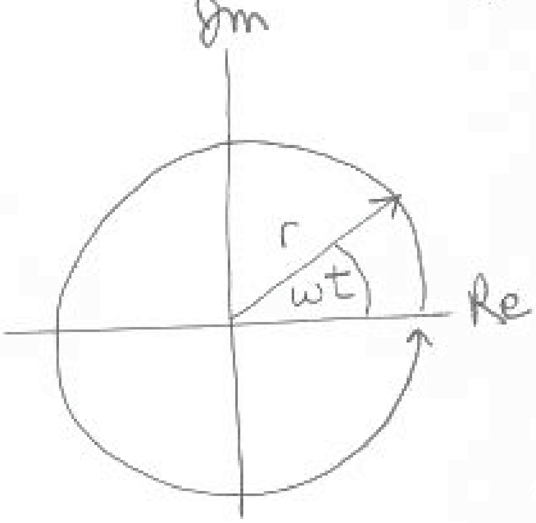
$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

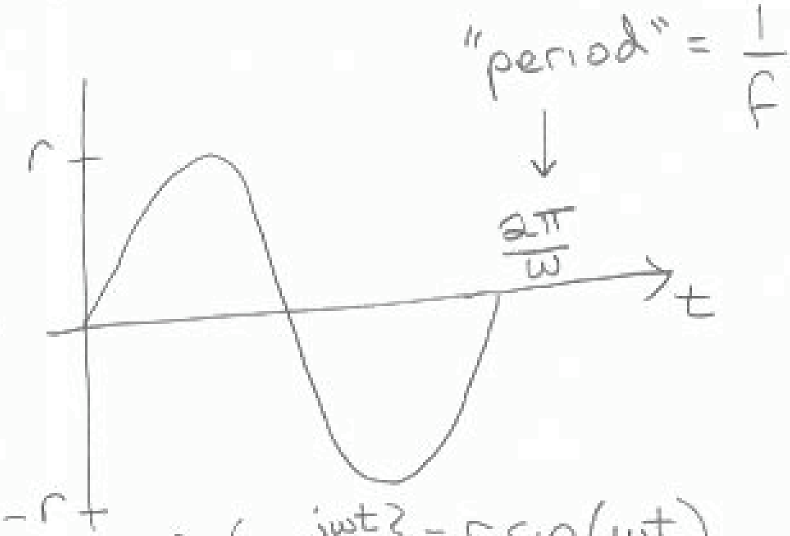
T: period, seconds/cycle

Now, make the phasor spin at $\omega = 2\pi f$

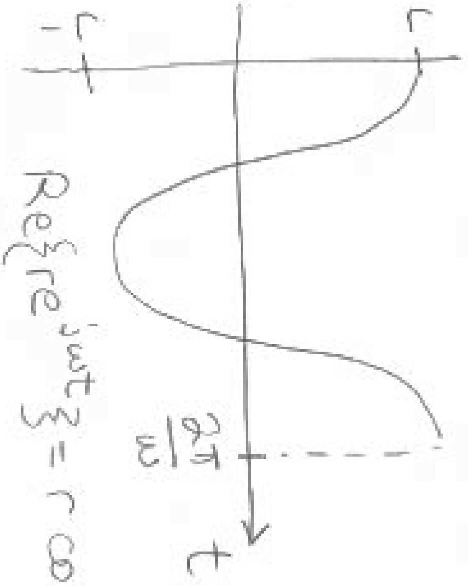
$$\underbrace{r}_{\text{amplitude}} e^{j\overbrace{\omega t}^{\text{frequency}}} = r\cos(\omega t) + j\,r\sin(\omega t)$$

$\boxed{\text{spinning arrow in the complex plane}}$



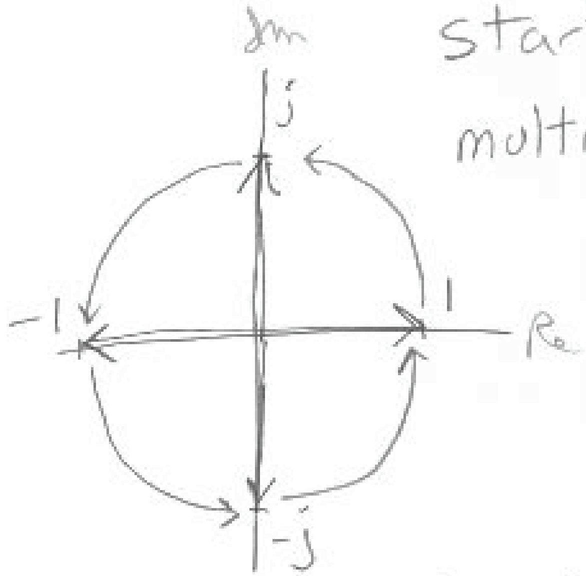


$$\mathrm{Im}\{re^{j\omega t}\} = \underbrace{r\sin(\omega t)}_{\text{note that this is a real number}}$$



$$\mathrm{Re}\{re^{j\omega t}\} = r\cos(\omega t)$$

QUADRATURE, COMPLEX NUMBERS



start with 1

multiply by j, you get $j, -1, -j, 1$

shift by $+90° = +\frac{\pi}{2}$

phasors do the same thing when you take their derivative:

$$\frac{de^{jt}}{dt} = j e^{jt} \qquad (\omega = 1)$$

as it turns out,
<u>any</u> complex number Z is
rotated by 90° when
multiplied by j.
We'll show you that in a minute.

$$\frac{d^2 e^{jt}}{dt} = j \cdot j e^{jt} = -e^{jt}$$

just like sinusoid,
and Hooke's Law