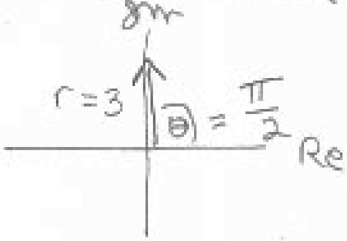
## Sample problems

convert the following complex numbers to cartesian coordinates $x + jy$ drawing a picture in the complex plane

① $3e^{j\frac{\pi}{2}}$




$x = 0$ , $y = 3$

$z = 0 + j3$

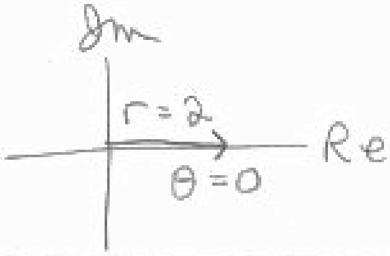
---

② $-2e^{-j3\pi} = \underbrace{-2e^{-j\pi}}_{\text{since } e^{j\theta} = e^{j(\theta + 2\pi k)}} = \underbrace{2e^{0}}_{\text{since } e^{j\theta} = -e^{j(\theta \pm \pi)}} = 2 + j0$

$k = 0, \pm 1, \pm 2, \ldots$

$x = 2$

$y = 0$

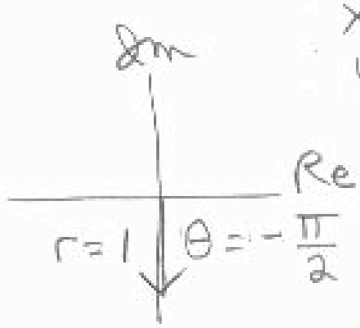



---

③ $je^{j\pi} = e^{j\frac{\pi}{2}}e^{j\pi} = e^{j\frac{3\pi}{2}} = e^{-j\frac{\pi}{2}} = 0 - j$

$x = 0$

$y = -1$



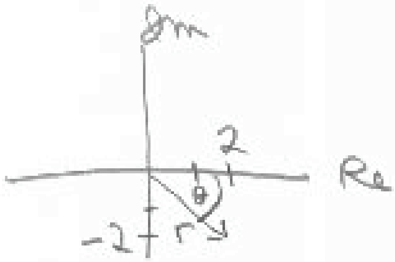
Convert the following complex numbers to polar coordinates $re^{j\theta}$

$r \geq 0 \quad -\pi < \theta \leq \pi$

① $\boxed{2 - 2j}$

$x = 2$

$y = -2$




$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

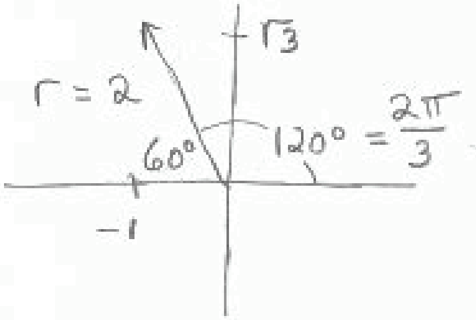
$\theta = -45^\circ = -\frac{\pi}{4}$

therefore, $2 - 2j = \boxed{2\sqrt{2}\, e^{-j\frac{\pi}{4}}}$

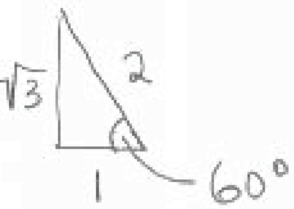
② $\boxed{-1 + \sqrt{3}\, j}$

$x = -1$

$y = \sqrt{3}$




$\boxed{2e^{j\frac{2\pi}{3}}}$



phase of a "phasor" is very handy

multiplication by a complex number

$$z = x + jy = re^{j\theta}$$

rotates by $\theta$ and scales by $r$

For example

$$(x_1 + jy_1)(x_2 + jy_2) =$$

$$(r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) =$$

$$r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

the first complex number has been scaled and rotated by the second.

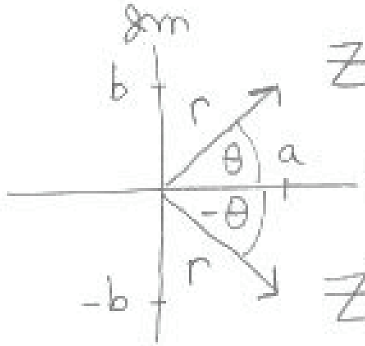
<u>NOTE</u>: phasors are not vectors. You can't multiply vectors by each other.

Now, make the second complex number

$$j = \underset{\uparrow r}{1} e^{j\overset{\theta}{\frac{\pi}{2}}}$$

Multiplying by $j$ rotates any complex number by $90°$

## Complex conjugates - reflect across x-axis




$$z = a + bj = re^{j\theta}$$

asterix means complex conjugate

$$z^* = a - bj = re^{-j\theta}$$ ← negative sign

### product

$$(a+bj)(a-bj) = a^2 + b^2 = r^2$$

or with phasors, phase cancels out

$$(re^{j\theta})(re^{-j\theta}) = r^2 e^{j(\theta-\theta)} = r^2 e^0 = r^2$$

$$z\, z^* = |z|^2$$ ← "modulus"

{ } not an algebraic operation.

### getting Re{z} and Im{z} using algebra

$$\mathrm{Re}\{z\} = a = \frac{(a+bj)+(a-bj)}{2} = \frac{z+z^*}{2}$$

$$\mathrm{Im}\{z\} = b = \frac{(a+bj)-(a-bj)}{2j} = \frac{z-z^*}{2j}$$

↑

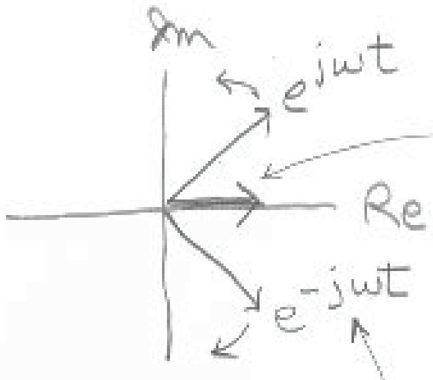
The imaginary coordinate b is itself real.

$$z = a + bj$$ ← this is imaginary.

Let's get algebraically correct –
algebra can't use $\mathrm{Re}\{\}$ or $\mathrm{Im}\{\}$
(in general, algebra doesn't use the squiggly bracket).

We, as engineers, eventually want real numbers.

---



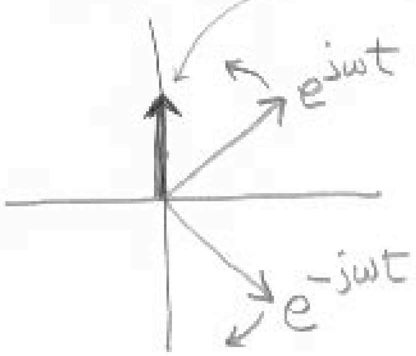

complex conjugates

$$\cos(\omega t) = \mathrm{Re}\{e^{j\omega t}\} = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$\boxed{\text{to get cos: subtract off imaginary part}}$

negative frequency,
backwards spinning phasor, complex conjugate

---




$$\sin(\omega t) = \underbrace{\mathrm{Im}\{e^{j\omega t}\}}_{\text{this is a real number so must divide by } j} = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$\boxed{\text{to get sin: subtract off real part.}}$

Now you can prove them.

example:

proove that $\boxed{\sin^2 t = \frac{1}{2}(1 - \cos 2t)}$

$$\sin t = \frac{e^{jt} - e^{-jt}}{2j} \qquad \omega = 1$$

$$\sin^2 t = \left(\frac{e^{jt} - e^{-jt}}{2j}\right)\left(\frac{e^{jt} - e^{-jt}}{2j}\right)$$

$$= \frac{e^{j2t} - e^{0} - e^{0} + e^{-j2t}}{-4}$$

$$= \frac{2\cos 2t - 2}{-4} = \frac{1}{2}(1 - \cos 2t)$$

since

$$\cos t = \frac{e^{jt} + e^{-jt}}{2}$$

remember the end of trig?

$$\sin 2t = 2 \sin t \cos t$$

$$\cos 2t = \cos^2 t - \sin^2 t$$

$$\sin \frac{1}{2} t = \pm \sqrt{\frac{1}{2}(1 - \cos t)}$$

you had to take them
on faith...
No longer!

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

in general we don't like imaginary or complex denominators. we get rid of them by multiplying by the complex conjugate

How to eliminate a complex denominator:

$$\frac{1}{a+jb} = \frac{1}{a+jb}\,\frac{a-jb}{a-jb} = \underbrace{\frac{a}{a^2+b^2}}_{\text{real part}} - \underbrace{\frac{jb}{a^2+b^2}}_{\text{imaginary part}}$$

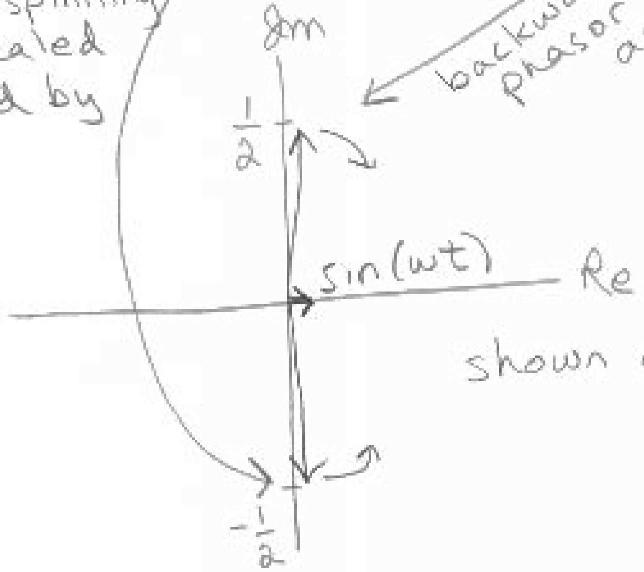
complex conjugate of $j$ is $-j$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \cdot \frac{-j}{-j} = -\frac{1}{2}j\,e^{j\omega t} + \frac{1}{2}j\,e^{-j\omega t}$$

forward spinning phasor scaled and rotated by $\underbrace{-\frac{1}{2}j}_{}$

magnitude and phase at $t=0$

backward spinning phasor scaled and rotated by $\frac{1}{2}j$




shown at $t=0$

any sinusoid can be represented by

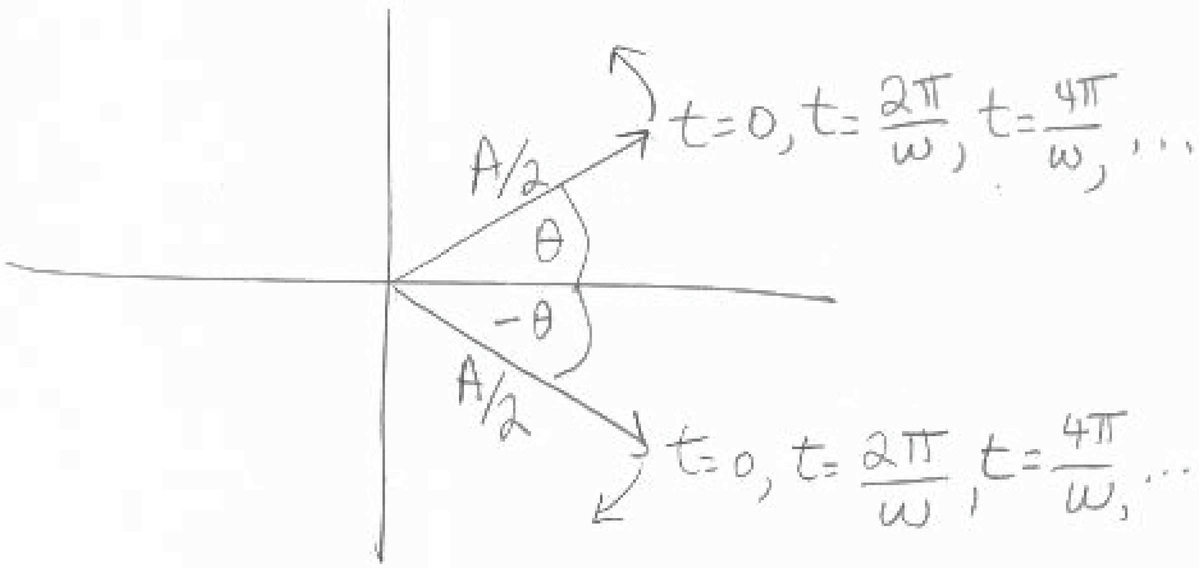
$$\underset{\uparrow\ \text{constant}}{A}\cos(\underbrace{\omega t+\theta}_{\text{substitute into equation for cos}}) = A\,\frac{e^{j(\omega t+\theta)}+e^{-j(\omega t+\theta)}}{2} =$$

complex conjugates

$$\overbrace{\frac{A}{2}e^{j\theta}e^{j\omega t} + \frac{A}{2}e^{-j\theta}e^{-j\omega t}}$$

amplitude   phase   frequency

a single complex number and its complement

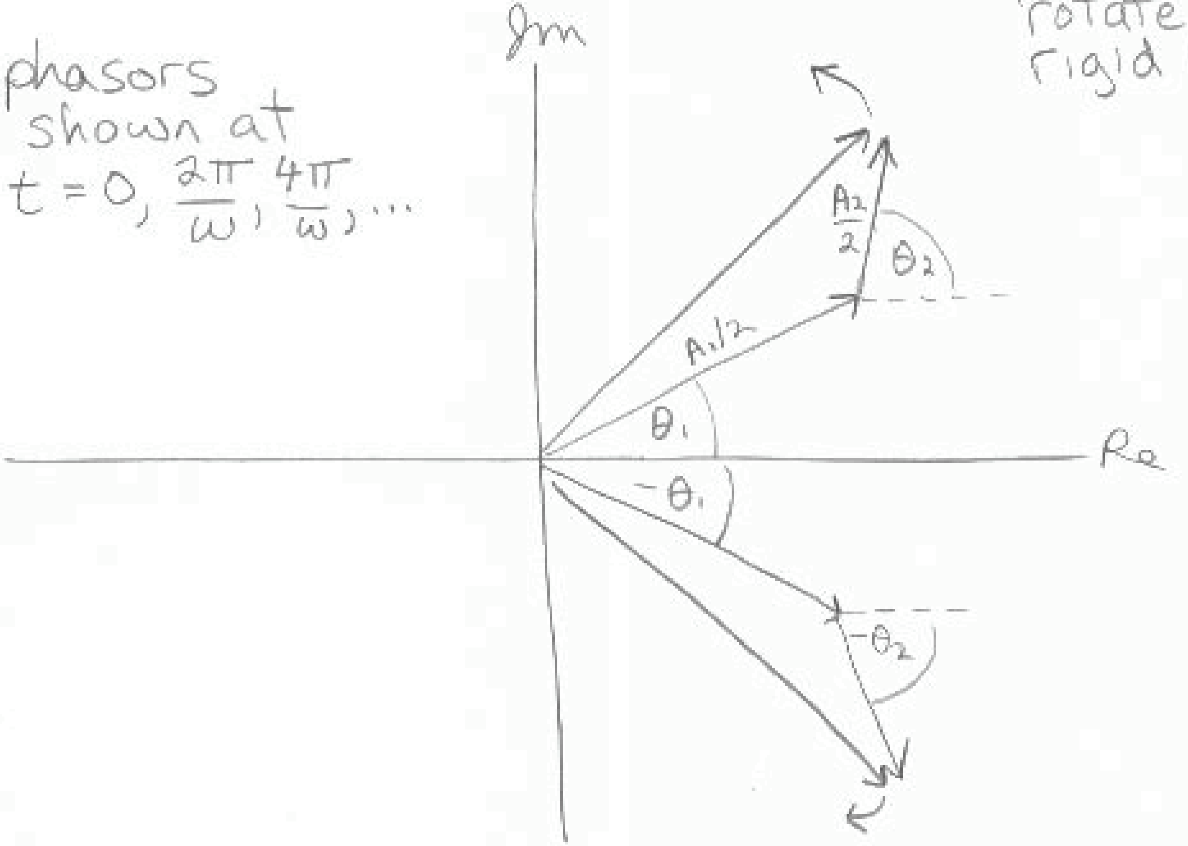



It's the sin & cos in one neat little package

# mystery of superposition revealed.

sum of same-frequency phasors rotate as a rigid unit.

phasors shown at $t = 0, \frac{2\pi}{\omega}, \frac{4\pi}{\omega}, \ldots$



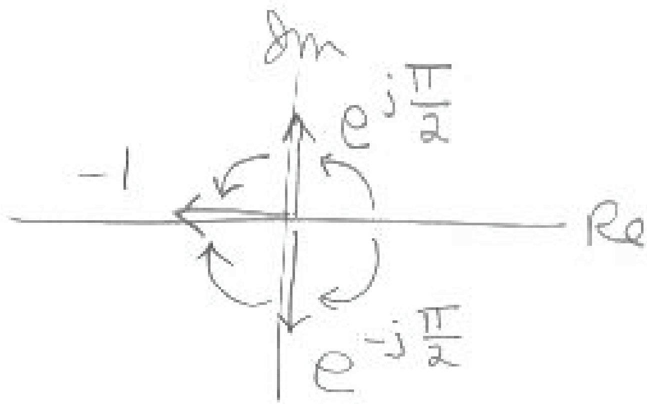

$$A_1 \cos(\omega t + \theta_1) + A_2 \cos(\omega t + \theta_2)$$

<u>IS</u> a sinusoid of frequency $\omega$

$$\underbrace{\left[\frac{A_1}{2} e^{j\theta_1} + \frac{A_2}{2} e^{j\theta_2}\right]}e^{j\omega t} + \underbrace{\left[\frac{A_1}{2} e^{-j\theta_1} + \frac{A_2}{2} e^{-j\theta_2}\right]}e^{-j\omega t}$$

Single complex number and its complement completely describe its phase and amplitude.

What are the square roots of $-1$ viewed as phasors?




The square roots are half way to $-1$ in either direction.

$$\sqrt{-1} = \pm j$$

$$+j = e^{j\frac{\pi}{2}}, \quad (+j)^2 = \left(e^{j\frac{\pi}{2}}\right)^2 = e^{j\pi} = 1$$

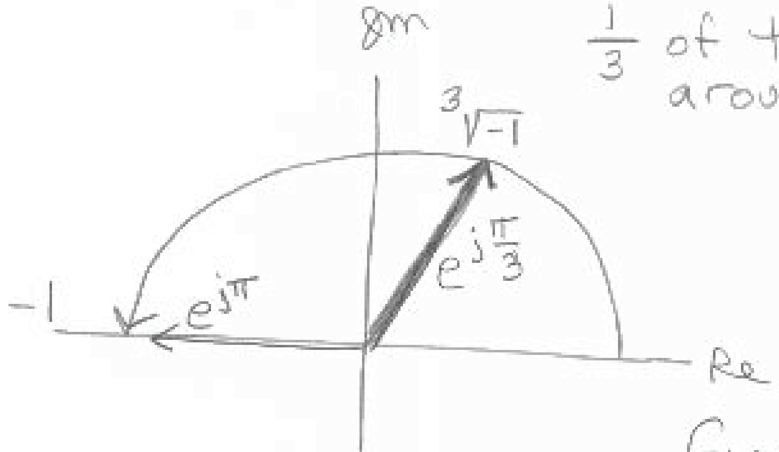
$$-j = e^{-j\frac{\pi}{2}}, \quad (-j)^2 = \left(e^{-j\frac{\pi}{2}}\right)^2 = e^{-j\pi} = 1$$

Now you can also answer intruiging questions like:

How do you find the cube root of $-1$ ?

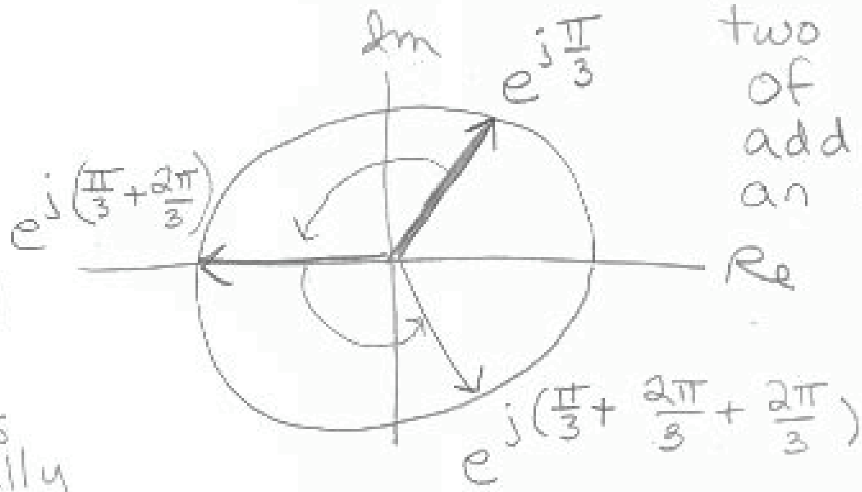
$$\text{Since } -1 = e^{j\pi} \quad \begin{cases} \theta = \pi \\ r = 1 \end{cases}$$

$$\sqrt[3]{-1} = e^{\frac{j\pi}{3}}$$




$\frac{1}{3}$ of the way around to $-1$

Find the other two cube roots of $-1$, by adding $\frac{1}{3}$ of an entire revolution, which, when cubed becomes an entire revolution, and thus disappears




this is actually $-1$, and indeed $(-1)^3 = -1$