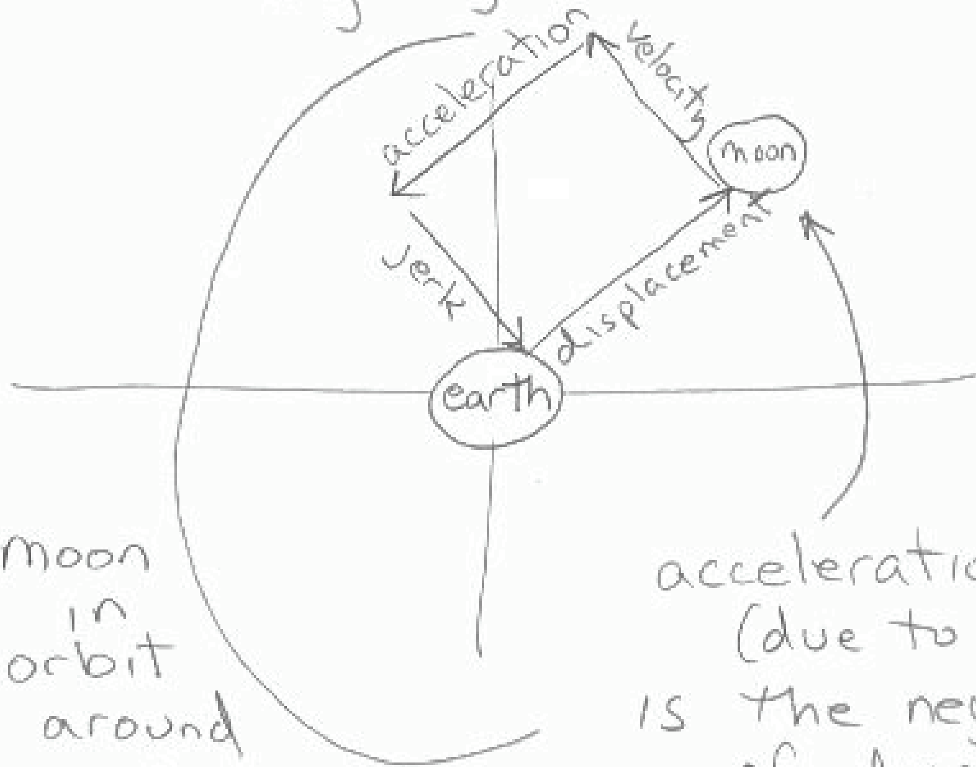
Here's another way to see that taking a derivative means shifting by 90°



moon in orbit around earth

acceleration (due to gravity) is the negative of displacement

like hookes law but in 2D.

$$\frac{d(re^{jwt})}{dt} = jw(re^{jwt})$$

rotated by 90° (j)

scaled by frequency (w)

phasors are great

"orthogonal basis functions"

or

"eigen functions"

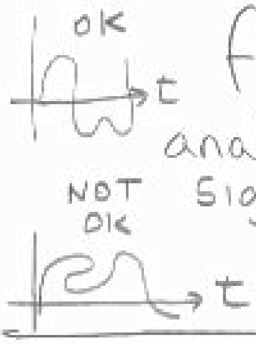
you can build many real (or complex) functions out of them. (But not all)

Linear combinations or derivatives of a given frequency phasor will only produce a phasor at that frequency.

hence, orthogonal

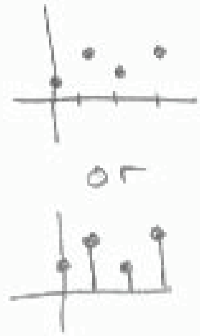
Signals

	continuous (real world)	discrete (sampled)
time	$f(t)$ analog signals	$f[n]$ digital signals n is an integer
space	$f(x, y \ldots)$ – optics, bioelectric fields, pressure/flow, antennae	$f[n, m \ldots]$ digital images finite element analysis computer graphics



A/D →

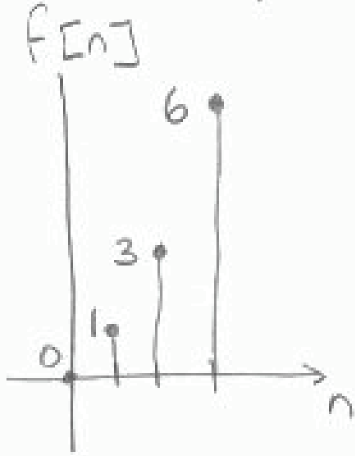
← D/A



discrete (sampled) vs "quantized", meaning function can only assume certain values.

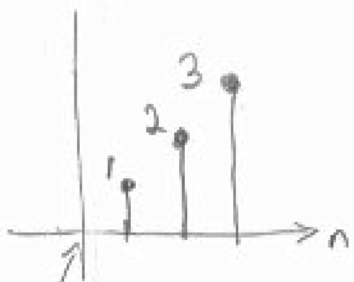
Discrete Differentiation

many choices, must define scale.
we will use "inner scale", simplest
definition: todays stock price
minus yesterday's.



$f[n]$

First Derivative

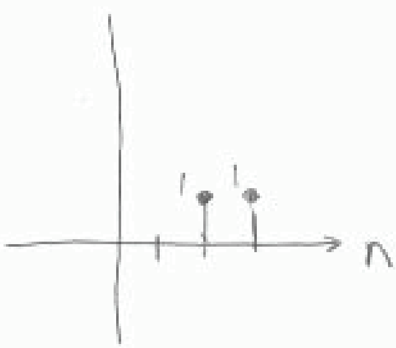


not defined at
$n=0$

$$g[n] = \frac{df[n]}{dn} = f[n] - f[n-1]$$

notice that dn is
assumed to be 1
since n is an integer

Second Derivative



not defined at
$n=0$ or $n=1$

$$\frac{d^2f[n]}{dn^2} = \frac{dg[n]}{dn} = \frac{df[n]}{dn} - \frac{df[n-1]}{dn} =$$

$$f[n] - f[n-1] - (f[n-1] - f[n-2]) =$$

$$f[n] - 2f[n-1] + f[n-2]$$

you need 2 terms to take a
first derivative and 3 terms
to take a second derivative.

Discrete Integration

The "running sum"

given

$$g[n] = \frac{df[n]}{dn} = f[n] - f[n-1]$$

$$f[n] = \sum_{k=-\infty}^{n} g[k]$$

as in continuous integration, we need a "dummy" (local) variable.

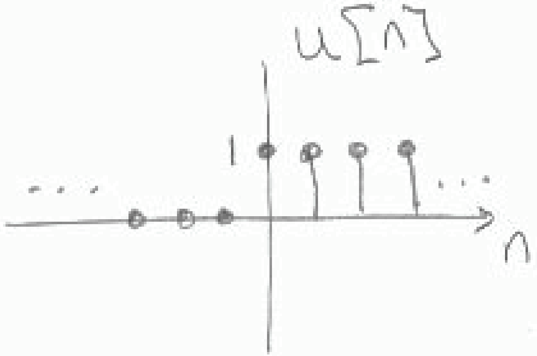
first you set n, then let k do its thing.

Integration and Differentiation are still opposites:

$$\frac{df[n]}{dn} = \sum_{k=-\infty}^{n} g[k] - \sum_{k=-\infty}^{n-1} g[k] = g[n]$$

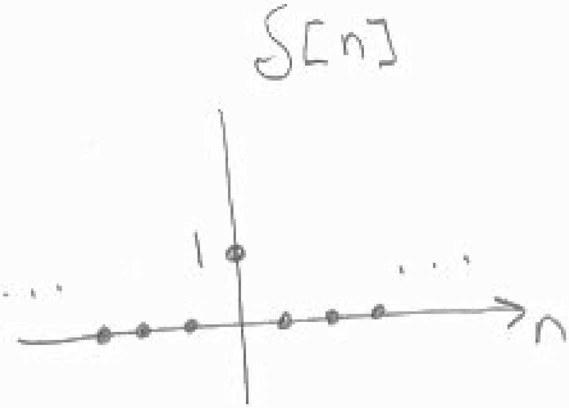
Now we introduce two new functions, and meet them first in the <u>discrete domain</u>

① The "Unit Step" function



$$u[n] \triangleq \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

② The "Impulse" function



$$\delta[n] \triangleq \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

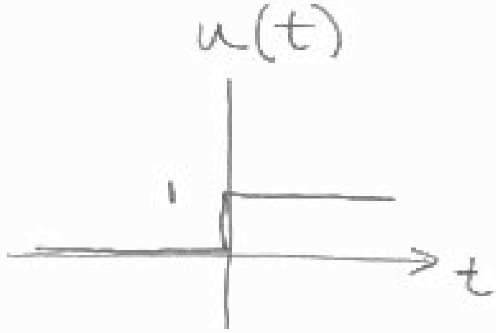
They are the derivative and integral of each other:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

$$\delta[n] = u[n] - u[n-1]$$

Now, in the continuous domain

① The "Unit Step" function



$$u(t) \triangleq \begin{cases} 1, & t > 0 \\ 0, & t < 0 \\ \text{not defined}, & t = 0 \end{cases}$$

② The "Impulse" function

drawn as an arrow of height 1

$$\delta(t) \triangleq \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

with the added stipulation that

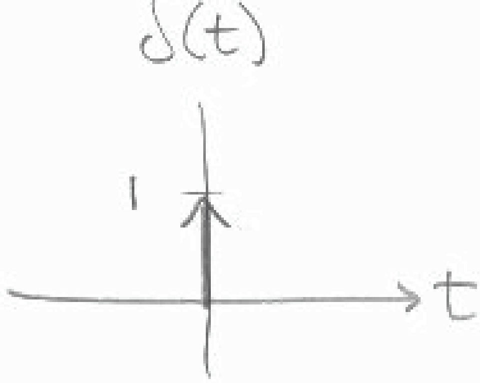
$$\int_{-\infty}^{+\infty} \delta(\tau)\, d\tau = 1$$

again, they are the derivative and integral of each other

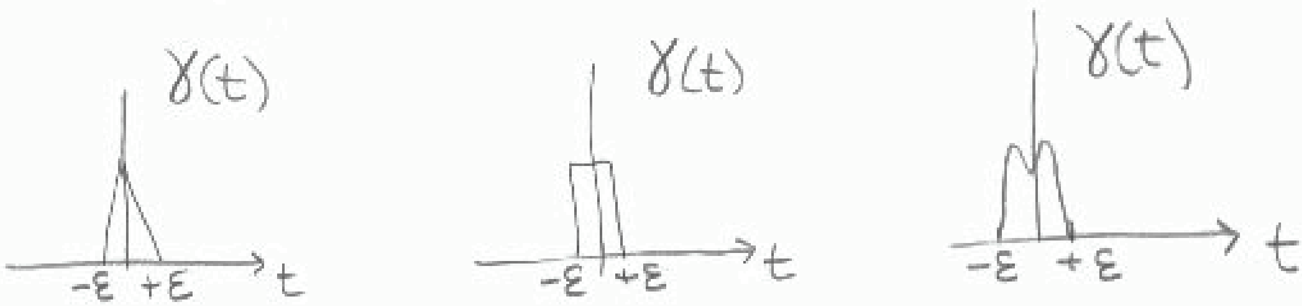
$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

$$\delta(t) = \frac{d\,u(t)}{dt}$$

the continuous impulse function



can be approximated by a very narrow pulse of any shape $\gamma(t) \cong \delta(t)$

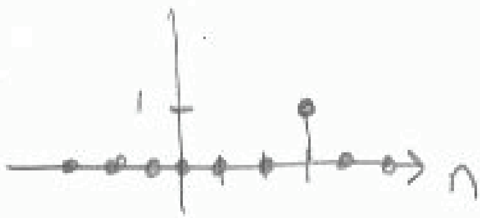


provided ε is small enough and

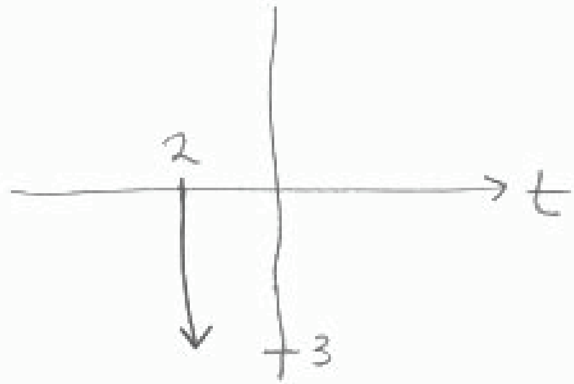
$$\int_{-\varepsilon}^{+\varepsilon} \gamma(\tau)\, d\tau = 1$$

examples of scaled and shifted impulse and unit step functions.

$\delta[n-3]$

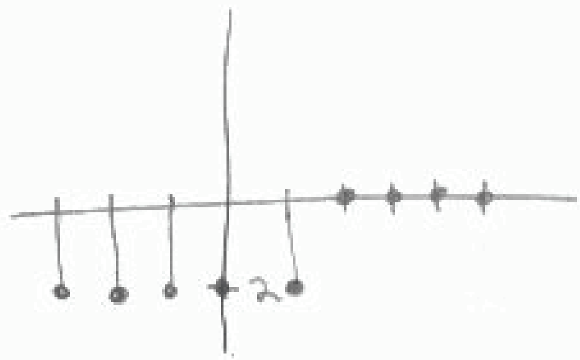


$-3\delta(t+2)$



$u(t-3)$

$-2u[1-n]$



can build any function $x[n]$
scaled and shifted impulse functions

$$x[n] = \sum_{k=-\infty}^{+\infty} a_k \delta[n-k]$$

they are orthogonal because they are zero at all other values of k