in general a_k is complex,
a stationary phasor

$$a_k = r_k e^{j\theta_k} \qquad r_k \text{ is real}$$

The k^{th} term of the Fourier series

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

is

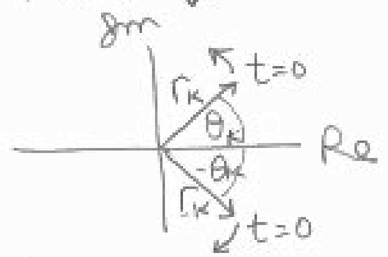
$$a_k e^{jk\omega_0 t} = \underbrace{r_k e^{j\theta_k}}_{\text{stationary}} \underbrace{e^{jk\omega_0 t}}_{\text{spinning}} = r_k e^{j(k\omega_0 t + \theta_k)}$$

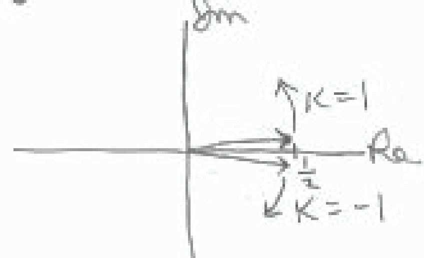
starts here at $t=0$

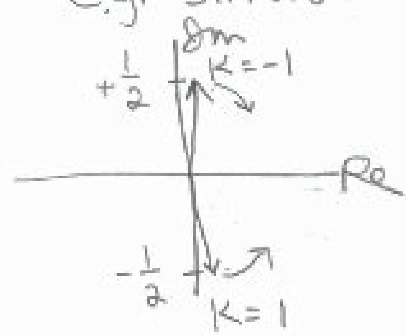
For real $x(t)$
The pair, k and $-k$ look like
this

e.g. $\cos \omega_0 t$

e.g. $\sin \omega_0 t$







2 phasors are complex conjugates.
one often sees the Fourier series written as

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} r_k \cos(k\omega_0 t + \theta_k)$$

$k=1$ ← only positive frequencies

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} \left[\mathrm{Re}\{a_k\} \cos(k\omega_0 t) - \mathrm{Im}\{a_k\} \sin(k\omega_0 t) \right]$$

$$a_{-k} = a_k^* \qquad a_0 \text{ must be real}$$

for real $x(t)$

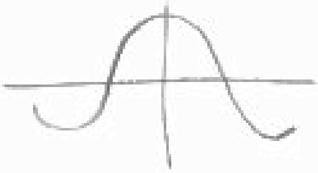
Time Reversal

$$x(-t) \overset{FS}{\longleftrightarrow} a_{-k} = a_k^*$$

for real $x(t)$ which is all we will be considering

FOR REAL $x(t)$

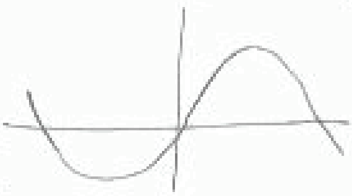
$Ev\{x(t)\}$



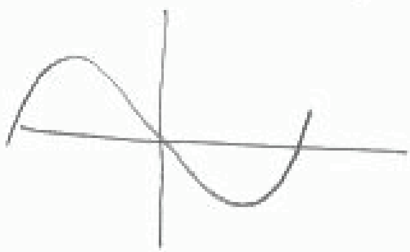
$Ev\{x(-t)\}$

For even parts of $x(t)$, real parts of a_k, $a_k = a_{-k}$

$Od\{x(t)\}$



$Od\{x(-t)\}$



For odd parts of $x(t)$, imag. parts of a_k $a_k = -a_{-k}$

that is why $a_{-k} = a^*$

For any $x(t)$, flipping to $x(-t)$ makes each phasor $e^{+jk\omega_0 t}$ go backwards, becoming $e^{-jk\omega_0 t}$.

k^{th} phasor becomes the $-k^{th}$ phasor

$$a_k \rightarrow a_{-k}$$

Differentiation

is linear, so you can add together the differentials of the components

$$\boxed{\frac{d\, e^{jk\omega_0 t}}{dt} = j\underbrace{k\omega_0}\, e^{jk\omega_0 t}}$$

90° phase shift

faster waves have steeper slopes

Differentiation boosts the high frequencies.

$$\boxed{\frac{dx(t)}{dt} \overset{Fs}{\longleftrightarrow} jk\omega_0 a_k}$$

Integration

like differentiation, linear, so operates independently on each component

first assume $a_0 = 0$, otherwise it integrates to infinity.... **No D.C.!**

$$\boxed{\int e^{jk\omega_0 t}\,dt = \frac{1}{jk\omega_0}e^{jk\omega_0 t}}$$

-90° phase shift

slower waves have more time to accumulate area.

Integration boosts the low frequencies

$$\boxed{\int x(t)\,dt \xleftrightarrow{FS} \frac{1}{jk\omega_0}a_k}$$

assumes $a_0 = 0$

constant delay: not constant phase.

$$x(t-\tau) \longleftrightarrow a'_k \quad \text{where } x(t) \longleftrightarrow a_k$$

$$e^{jk\omega_0(t-\tau)} = e^{-jk\omega_0\tau} e^{jk\omega_0 t}$$

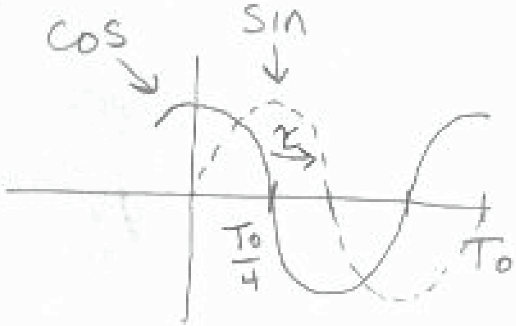
$$\boxed{a'_k = e^{-jk\omega_0\tau} a_k}$$

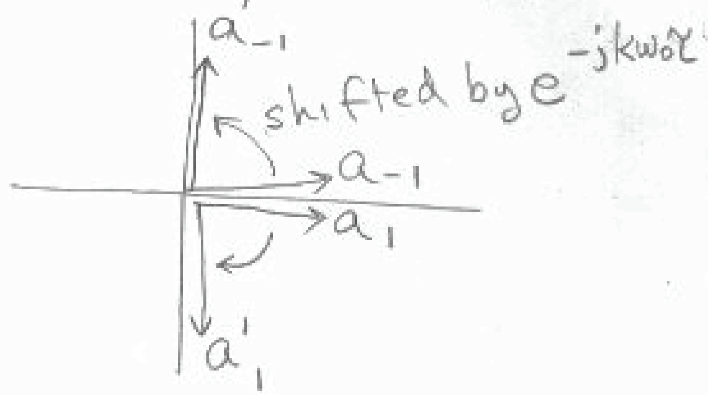
k examine each harmonic

0 DC $a'_0 = a_0$ shifting DC doesn't change it

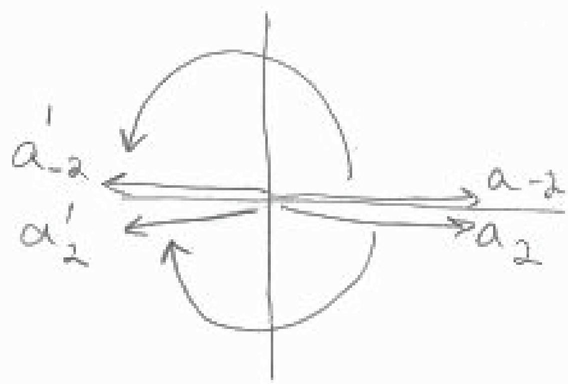
1 fundamental, eg pure cosine $a_1 = a_{-1} = 1/2$

$\tau = \frac{T_0}{4}$ where $T_0 = \frac{2\pi}{\omega_0}$ = period





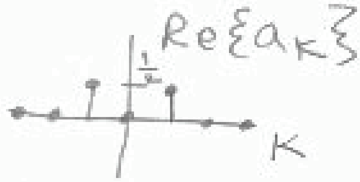
2 Second harmonic



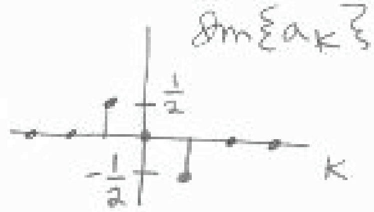
frequency $\left(\frac{\text{radians}}{\text{sec}}\right) \cdot$ time (sec) = phase (radians)

Fourier Series - review

$\cos \omega_0 t \quad \xleftrightarrow{Fs}$



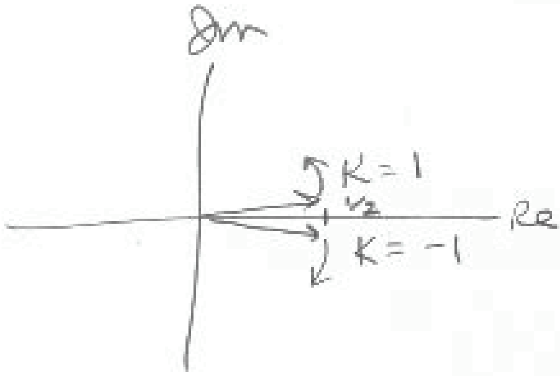
$\sin \omega_0 t \quad \xleftrightarrow{Fs}$

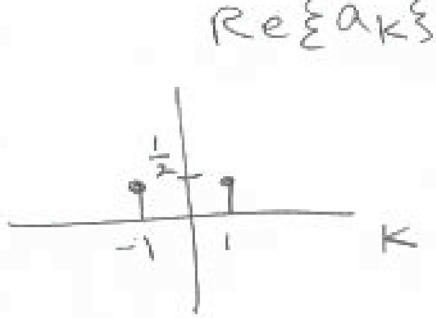


<u>synthesis</u> $\quad x(t) = \sum_{K=-\infty}^{+\infty} a_K e^{jK\omega_0 t}$

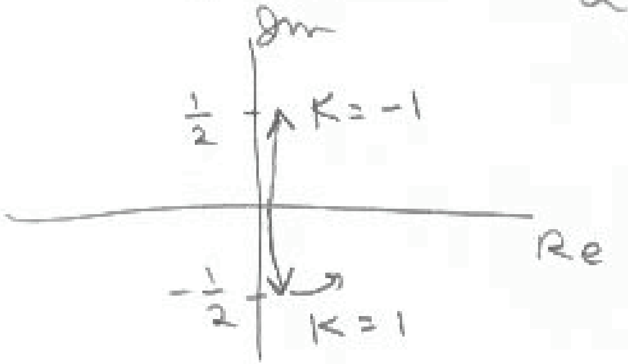
<u>analysis</u> $\quad a_K = \frac{1}{T_0} \int_{T_0} x(t) e^{-jK\omega_0 t} dt$

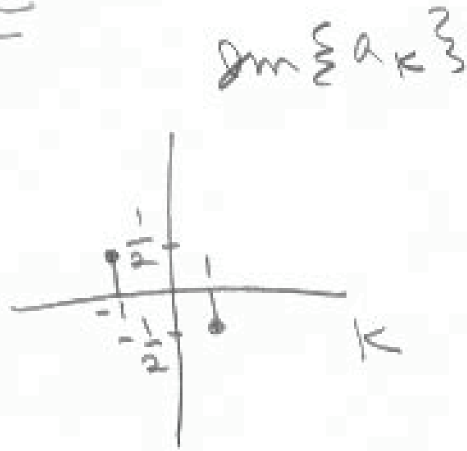
$$\underline{\cos \omega_0} t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$





$$\underline{\sin \omega_0} t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$





More complex example

$$x(t) = 1 + \underbrace{2\cos(\omega_0 t) + \sin(\omega_0 t)}_{\text{fundamental}} + \underbrace{\cos(\overset{\text{frequency}}{2\omega_0} t + \overset{\text{phase}}{\tfrac{\pi}{4}})}_{\text{2nd harmonic}}$$

$a_0 = 1$ "DC"

$$\boxed{x(t) = \sum_{-\infty}^{+\infty} a_k e^{jk\omega_0 t}}$$

$a_1 = ?$ $(k=1)$

$a_{-1} = ?$ $(k=-1)$

$$\text{fundamental} = \underbrace{\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]}_{\text{cos term}} + \underbrace{\frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]}_{\text{sin term}}$$

$$a_1 = 1 + \frac{1}{2j} = 1 - \frac{1}{2}j$$

$$a_{-1} = 1 - \frac{1}{2j} = 1 + \frac{1}{2}j$$

complex conjugates

$a_2 = ?$ $(k=2)$

$a_{-2} = ?$ $(k=-2)$

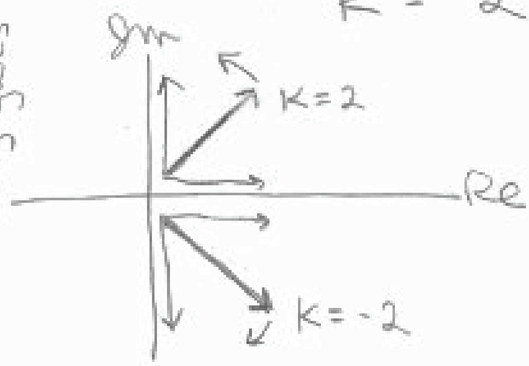
$$\text{2nd harmonic} = \frac{1}{2}\left[e^{j(2\omega_0 t + \frac{\pi}{4})} + e^{-j(2\omega_0 t + \frac{\pi}{4})}\right]$$

$$= \underbrace{\frac{1}{2}e^{j\frac{\pi}{4}}}_{k=2} e^{j2\omega_0 t} + \underbrace{\frac{1}{2}e^{-j\frac{\pi}{4}}}_{k=-2} e^{-j2\omega_0 t}$$

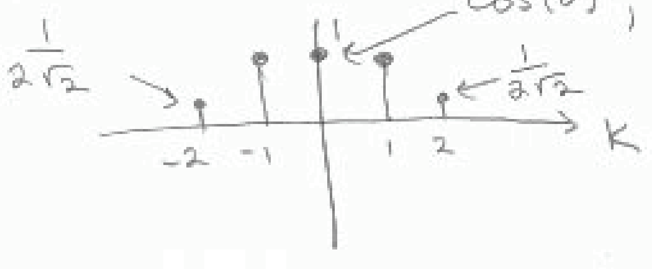
$$a_2 = \frac{1}{2}e^{j\frac{\pi}{4}} = \frac{1}{2\sqrt{2}}(1+j)$$

$$a_{-2} = \frac{1}{2}e^{-j\frac{\pi}{4}} = \frac{1}{2\sqrt{2}}(1-j)$$

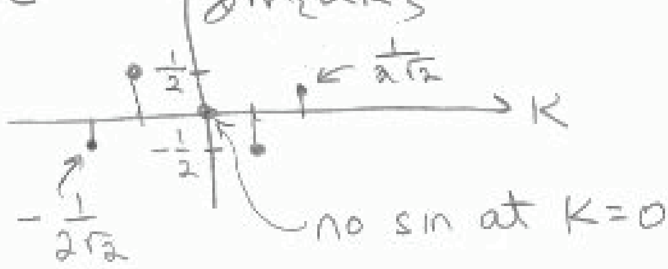
complex conjugates



$Re\{a_k\}$



$Im\{a_k\}$



Fourier Series example - Synthesis
Real coefficients a_k

$a_0 = 1$

$a_1 = a_{-1} = 2$

$a_2 = a_{-2} = 3$

$a_k = 0$, otherwise

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}}$$

Lets say $\omega_0 = 2\pi \frac{\text{radians}}{\text{sec}}$

$\left(f_0 = 1 \frac{\text{cycle}}{\text{sec}}\right)$

hear the choir each time!!

$$x(t) = \underbrace{1}_{k=0} + 2(\underbrace{e^{j2\pi t}}_{1} + \underbrace{e^{-j2\pi t}}_{-1}) + 3(\underbrace{e^{j4\pi t}}_{2} + \underbrace{e^{-j4\pi t}}_{-2})$$

$$= 1 + \underbrace{4\cos(2\pi t)}_{\text{fundamental or "first" harmonic}} + \underbrace{6\cos(4\pi t)}_{\text{second harmonic}}$$