Fourier Transform

synthesis

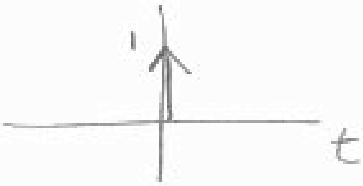
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

analysis

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

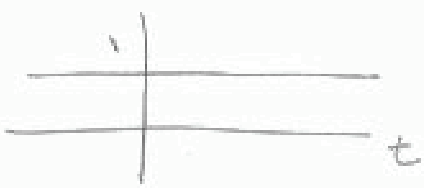
Examples of Fourier Transform

$$x(t) = \delta(t) \xleftrightarrow{F} X(\omega) = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\omega t}\, dt = 1$$



white noise
"rain drops" or "wind molecules"

$$x(t) = 1 \longleftrightarrow X(\omega) = 2\pi\delta(\omega)$$



this is periodic, and the impulse in the fourier transform is the equivalent of $a_0 = 1$ in the fourier series.
we need the 2π to make the synthesis equation work

$$x(t) = 1 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{2\pi\delta(\omega)}_{X(\omega)}\, e^{j\omega t}\, d\omega$$

In general, for any phasor

$$a_k e^{jk\omega_0 t}$$

in a periodic signal, the

$$a_k$$

term in the Fourier Series becomes

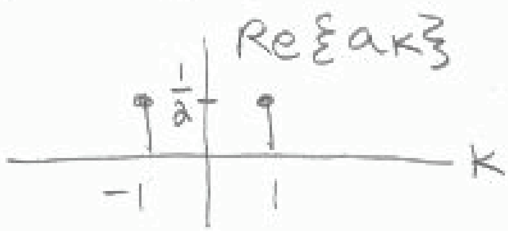
$$a_k 2\pi \delta(\omega - k\omega_0)$$

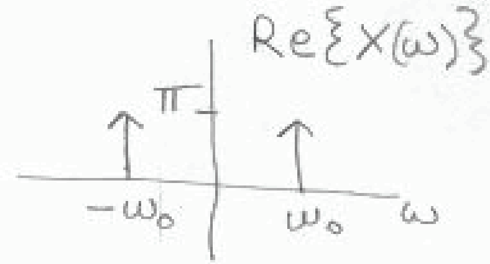
in the Fourier Transform

so, for example,

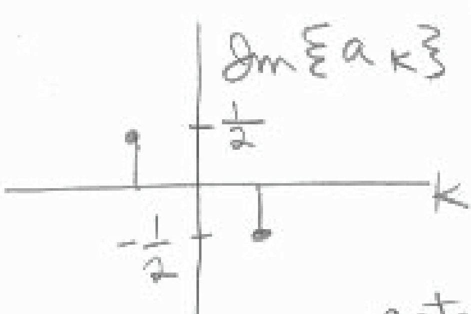
Fourier Series | Fourier Transform

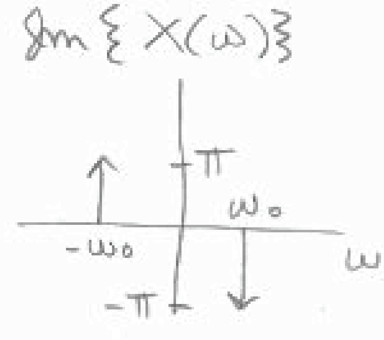
$$\frac{1}{2}\delta[k-1] + \frac{1}{2}\delta[k+1] \longleftrightarrow \cos(\omega_0 t) \overset{F}{\longleftrightarrow} \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$





$$-\frac{1}{2}j\,\delta[k-1] + \frac{1}{2}j\,\delta[k+1] \overset{Fs}{\longleftrightarrow} \sin(\omega_0 t) \overset{F}{\longleftrightarrow} -\pi j\,\delta(\omega-\omega_0) + \pi j\,\delta(\omega+\omega_0)$$





note: impulses can be imaginary

for any periodic Signal

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \overset{F}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$$

scaled by 2π

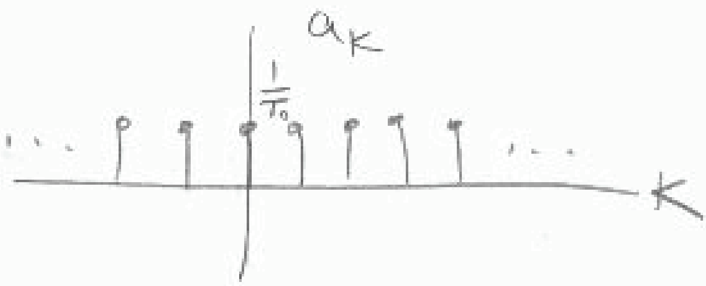
now impulses instead of finite values a_k

now a function of ω instead of k

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_0)$$

$$\omega_0 = \frac{2\pi}{T_0}$$

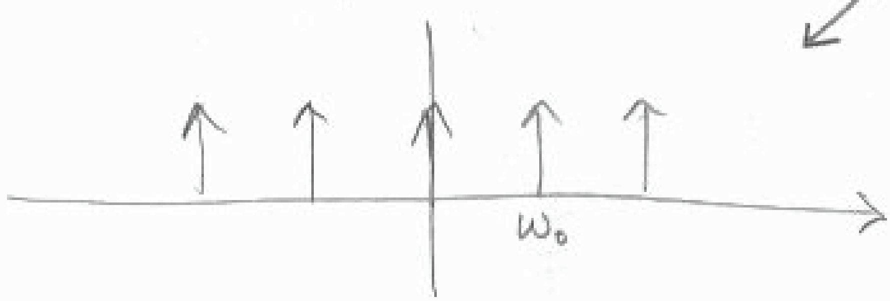
$$a_K = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} \delta(t) e^{-jk\omega_0 t} \, dt = \frac{1}{T_0}$$



periodic signals have finite power but infinite energy

since $X(\omega) = \sum_{K=-\infty}^{+\infty} 2\pi a_K \delta(\omega - K\omega_0)$ for any periodic signal

$$X(\omega) = \frac{2\pi}{T_0} \sum_{K=-\infty}^{+\infty} \delta(\omega - K\omega_0)$$

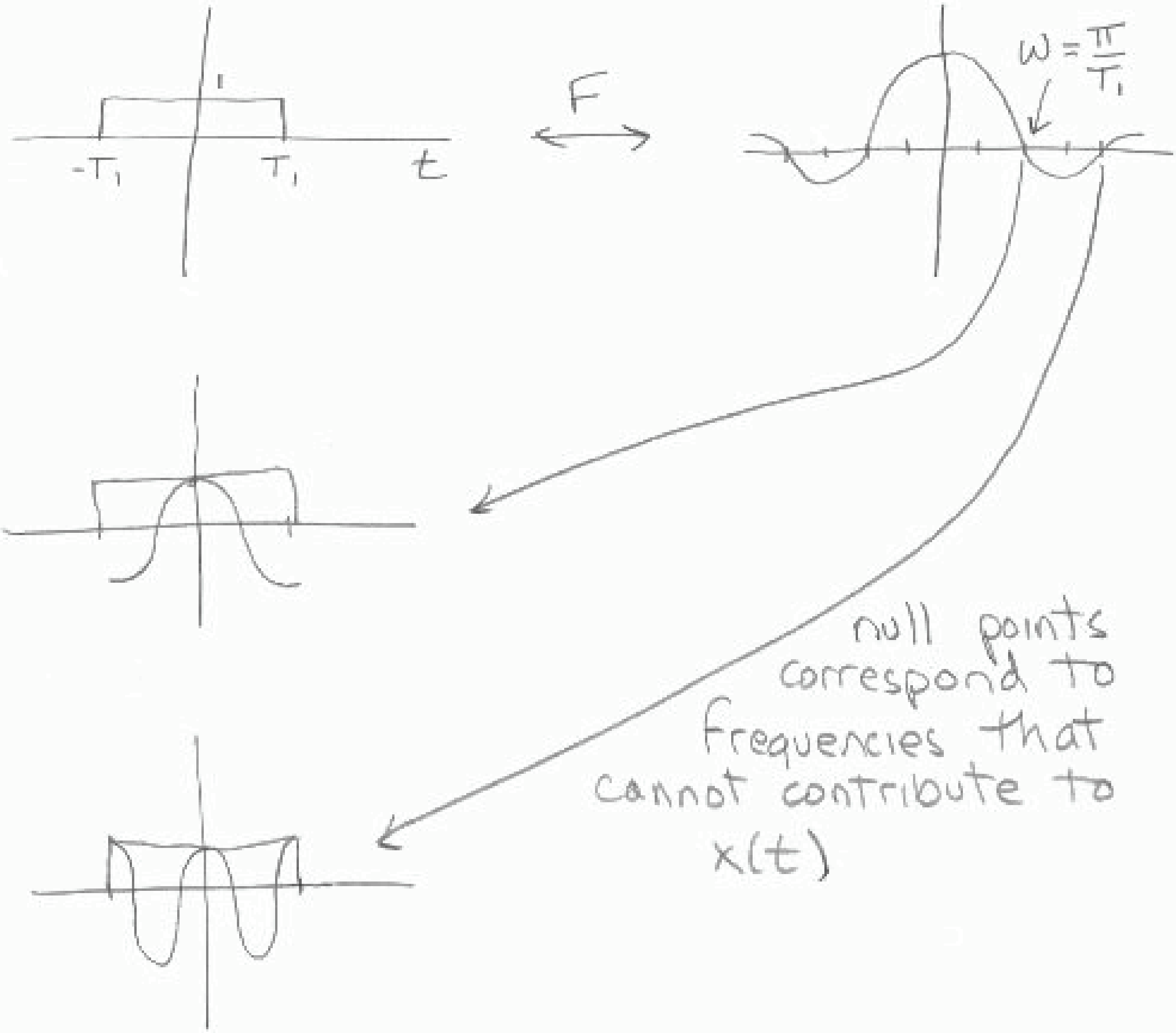


Fourier Transform has "boosted the gain" to see signals with <u>finite</u> energy, i.e. non-period signals

Example

"sinc" function

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \overset{F}{\longleftrightarrow} \frac{2 \sin \omega T_1}{\omega}$$



Like the Fourier Series, the Fourier Transform is

linear $\quad x(t) \xleftrightarrow{F} X(\omega)$ $\quad$ a, b are real or complex

$$y(t) \xleftrightarrow{F} Y(\omega)$$

$$a x(t) + b y(t) \longleftrightarrow a X(\omega) + b Y(\omega)$$

time shifting $\quad x(t-\tau) \xleftrightarrow{F} e^{-j\omega\tau} X(\omega)$

time reversal $\quad x(-t) \xleftrightarrow{F} X(-\omega)$

time scaling $\quad x(\alpha t) \xleftrightarrow{F} \frac{1}{\alpha} X\left(\frac{\omega}{\alpha}\right)$ $\quad$ real $\alpha > 0$

proof:

let $\tau = \alpha t$

$d\tau = \alpha\, dt$

$$\int_{-\infty}^{+\infty} x(\alpha t) e^{-j\omega t}\, dt = \frac{1}{\alpha} \int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/\alpha)\tau}\, d\tau$$

(note: different from Fourier Series, which does not change with scaling)

differentiation

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(\omega)$$

integration, assuming $X(0) = 0$ No D.C.

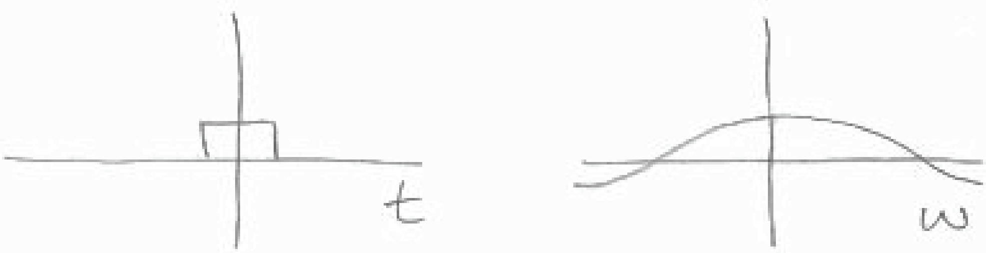
$$\int x(\tau)\, d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(\omega)$$

more on time scaling

$$x(\alpha t) \overset{F}{\longleftrightarrow} \frac{1}{\alpha} X\left(\frac{\omega}{\alpha}\right)$$

$\alpha = \frac{1}{2}$
slowing down $x(t)$ compresses spectrum to lower ω's

$\alpha \to 0$



$\alpha = 2$
speeding up $x(t)$ spreads spectrum to higher ω's

$\alpha \to \infty$

COMPLEX CONJUGATES

For Real $x(t)$

complex conjugate

$$X(-\omega) = X^{*}(\omega) \qquad (a+bj)^{*} = (a-bj)$$

cosines $\quad \mathrm{Re}\{X(\omega)\} = \mathrm{Re}\{X(-\omega)\}$

sines $\quad \mathrm{Im}\{X(\omega)\} = -\mathrm{Im}\{X(-\omega)\}$

$\mathrm{Ev}\{x(t)\} \xleftrightarrow{F} \mathrm{Re}\{X(\omega)\}$ cosines

$\mathrm{Od}\{x(t) \xleftrightarrow{F} \mathrm{Im}\{X(\omega)\}$ sines

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~~CONVOLUTION~~

$$\cancel{x(t) * h(t) \xleftrightarrow{F} X(\omega)\, H(\omega)}$$

Convolution in time domain equals multiplication in frequency domain

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

$$y(t) = x(t) * h(t)$$

$$Y(\omega) = X(\omega)\,H(\omega)$$

The phasor in $x(t)$ at frequency ω is scaled and rotated by the phasor in $x(t)$ at frequency ω, to become a phasor in $y(t)$ <u>at frequency ω</u>. All that changes is magnitude and phase power/energy comes from ohms law

$$\frac{V}{R} = I$$

$$IR = V$$

force · distance/time

power $P = \underset{\text{force}}{V}\,\underset{\text{distance/time}}{I} = \frac{V^2}{R} = I^2 R$

energy $E = \int P\,dt$

how much gas is in the generator

instantaneous power

$P = \frac{dE}{dt}$

how bright is the light bulb

average power

$P_{\text{average}} = \frac{\Delta E}{\Delta t}$

how much gas was used over the intire hour.