convolution + phasors = Fourier

Let's see what happens when we put a single unit phasor through an LTI system

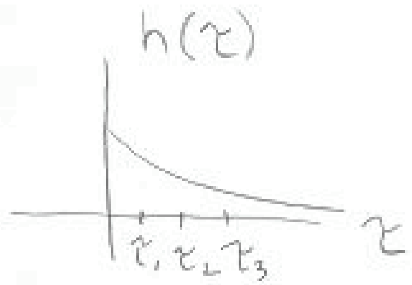
$$x(t) = e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) \qquad y(t) = x(t) * h(t)$$

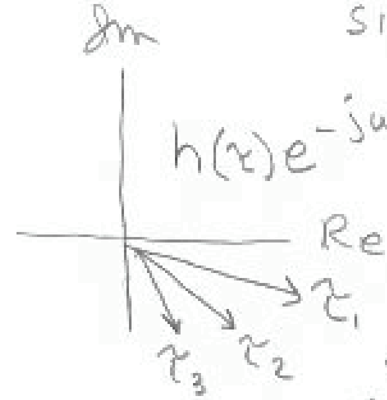
or, by commutative property

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\,x(t-\tau)\,d\tau \quad \longleftarrow \quad y(t) = h(t) * x(t)$$

Fourier Analysis!

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\,e^{j\omega(t-\tau)}\,d\tau = e^{j\omega t}\underbrace{\int_{-\infty}^{+\infty} h(\tau)\,e^{-j\omega\tau}\,d\tau}_{\text{single complex number } H(\omega)}$$





add these vectors to get $H(\omega)$

$$H(\omega) \overset{F}{\longleftrightarrow} h(t)$$

if $x(t) = e^{j\omega t}$

$$y(t) = e^{j\omega t} H(\omega)$$

Since any $x(t) = \int_{-\infty}^{+\infty} X(\omega)\,e^{j\omega t}\,d\omega$ is just the weighted sum of a bunch of phasors, therefore

$$Y(\omega) = X(\omega)\,H(\omega)$$

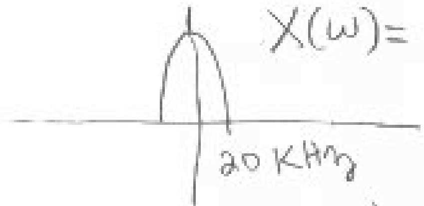
Each phasor in the output is a scaled and shifted version of the phasor in the input.

Multiplication

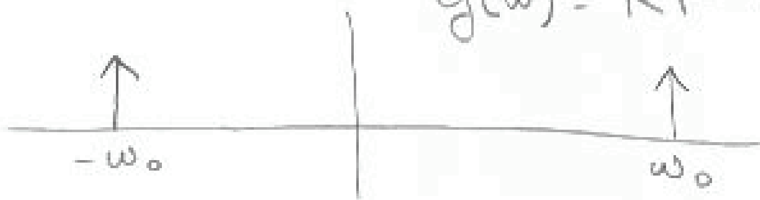
$$x(t)\,y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X(\omega) * Y(\omega)$$

Example
AM Radio

$X(\omega)$ = audio signal

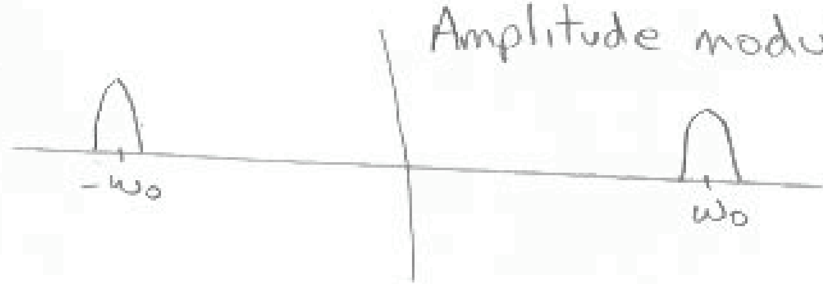


$y(\omega)$ = RF carrier cos(t)
at
$\omega_o = 100$ MHz

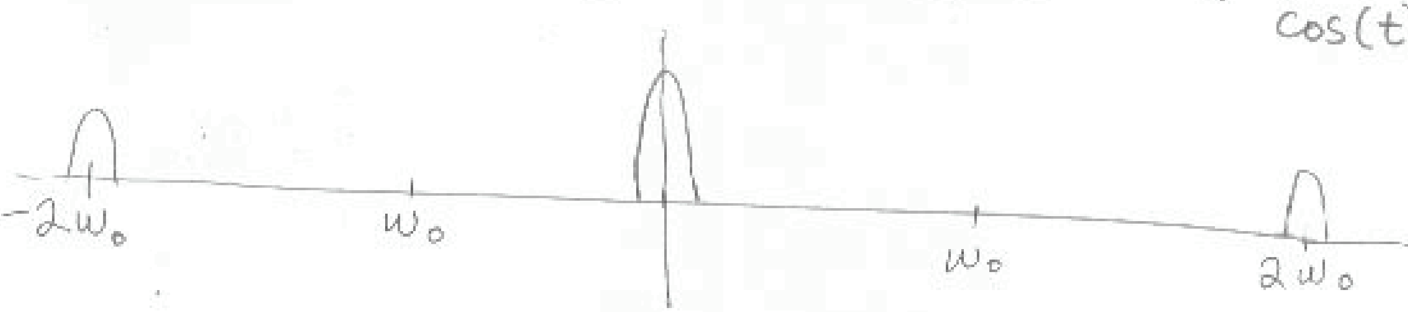


Amplitude modulation $x(t)y(t)$

Xmitted
Signal



Demodulate by multiplying again by $y(t) = \cos(t)$



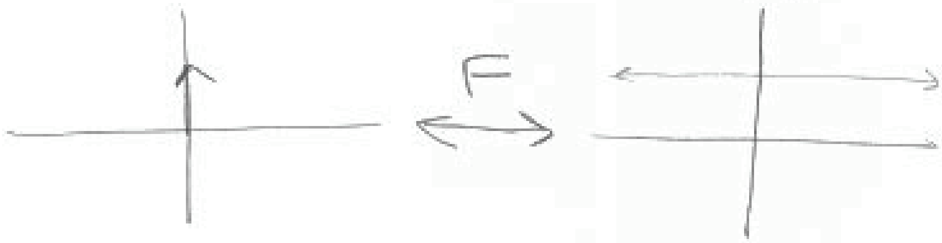
eg. first stage of MR scanner

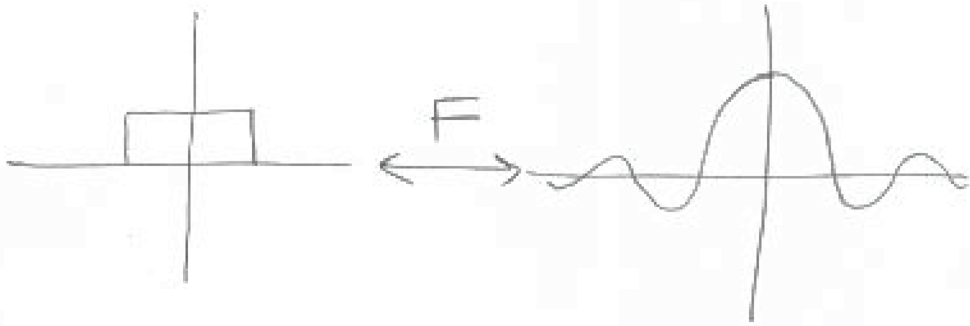
Duality Property

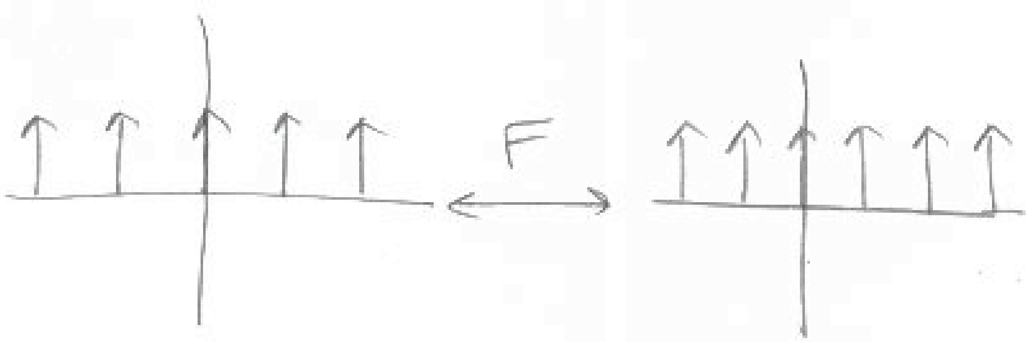
Schaum (eq 5.54)
(prob 5.18)

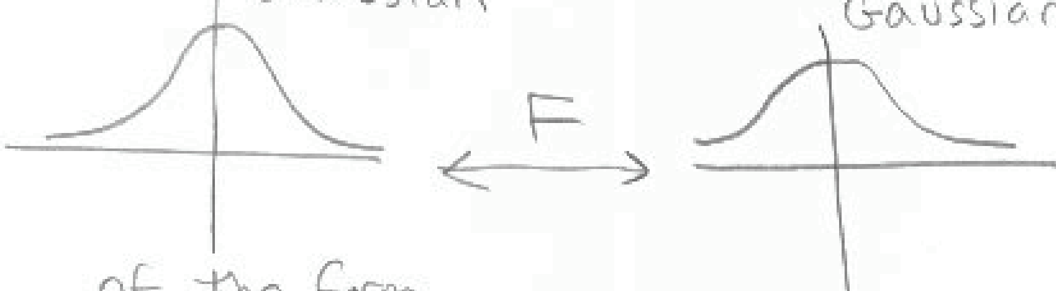
$$X(t) \xleftrightarrow{F} 2\pi x(-\omega)$$

so, except for coefficients
← Either $x(t)$ or $X(\omega)$ →







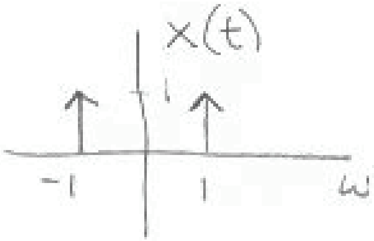


(prob 5.43)

of the form
$Ae^{-\frac{t^2}{\sigma^2}}$

EXAMPLE

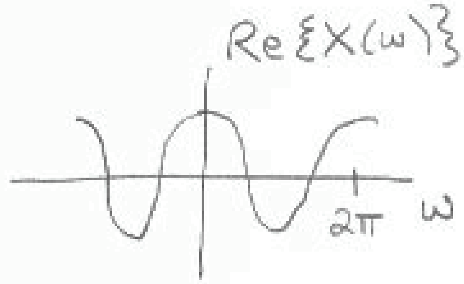
$x(t) = \delta(t-1) + \delta(t+1)$, what is $X(\omega)$?



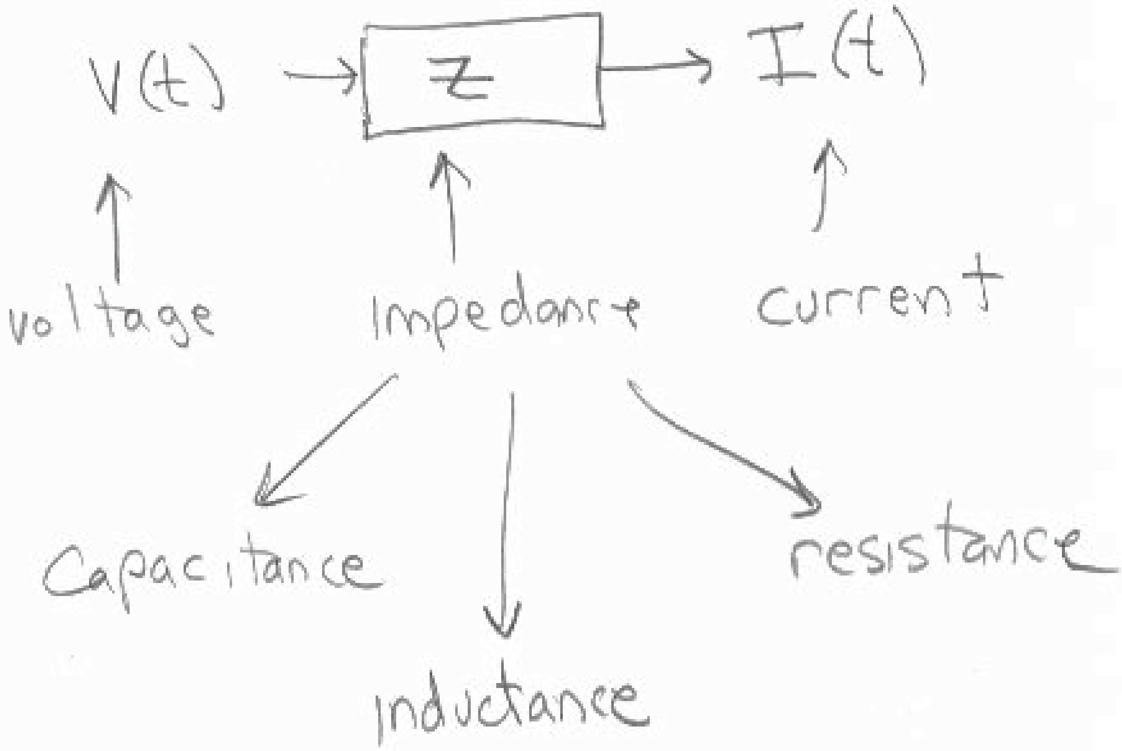
$$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(\omega) = \left[\int_{-\infty}^{+\infty} \delta(t-1)\, e^{-j\omega t}\, dt + \int_{-\infty}^{+\infty} \delta(t+1)\, e^{-j\omega t}\, dt \right]$$

$$X(\omega) = \left[e^{-j\omega} + e^{j\omega} \right] = 2\cos(\omega)$$



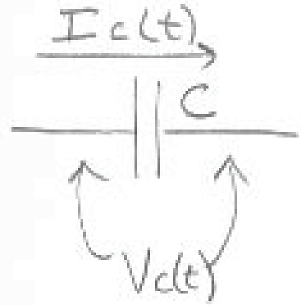
Now consider the electrical impedance of a compontent as an LTI system



Complex impedance Z

replaces resistance R (which is real)

$$I_c(t) = C\frac{dV_c(t)}{dt}$$



if $V_c(t) = e^{j\omega t}$

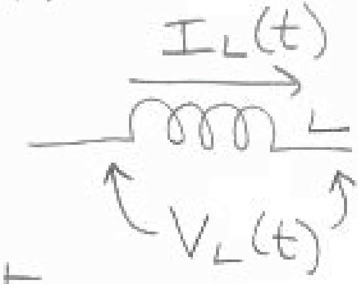
then $I_c = j\omega C e^{j\omega t}$

Complex Impedance using Ohms Law

$$Z_c = \frac{V_c(t)}{I_c(t)} = \frac{e^{j\omega t}}{j\omega C e^{j\omega t}} = \frac{1}{j\omega C} = -\frac{j}{\omega C}$$

Likewise for a coil

$$V_L(t) = L\frac{dI_L(t)}{dt}$$



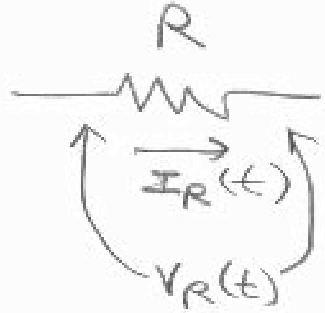
if $I_L(t) = e^{j\omega t}$

then $V_L = j\omega L e^{j\omega t}$

complex impedance

$$Z_L = \frac{V_L(t)}{I_L(t)} = \frac{j\omega L e^{j\omega t}}{e^{j\omega t}} = j\omega L$$

what is complex impedance of resistor



$$V_R(t) = e^{j\omega t}$$

$$I_R(t) = \frac{e^{j\omega t}}{R}$$

$$Z_R = R$$ pure real

Inductance or capacitance is called "reactance", pure imaginary impedance of circuits

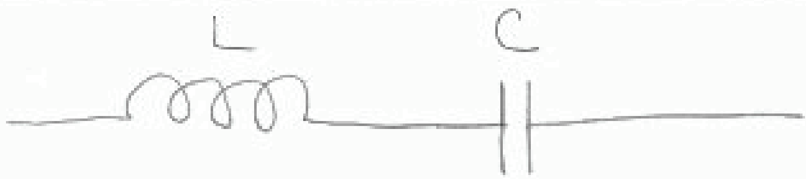
"complex" impedance of R is just a real number

time constant

$$Z = \frac{1}{j\omega C} + R = \frac{1 + j\omega \overbrace{RC}}{j\omega C}$$

$$Z = R \Big|_{\omega \gg \frac{1}{RC}}$$ "High Pass"

$$Z = \frac{1}{j\omega C} \Big|_{\omega \ll \frac{1}{RC}}$$

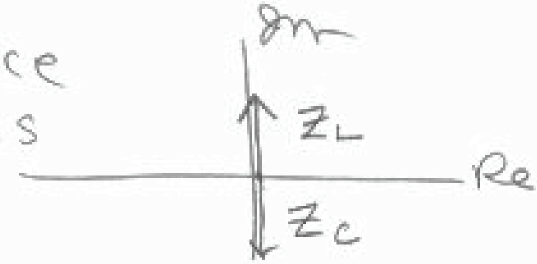


time constant

$$Z = j\omega L + \frac{1}{j\omega C} = \frac{1 - \omega^2 \overbrace{LC}}{j\omega C}$$

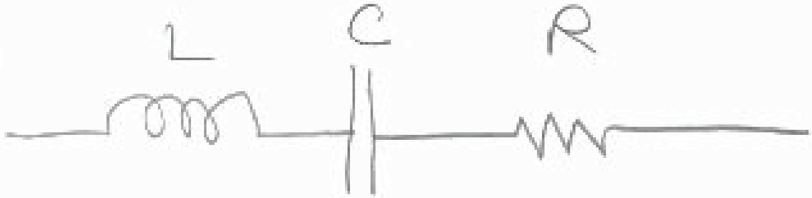
$$Z = 0 \Big|_{\omega = \frac{1}{\sqrt{LC}}}$$ Resonance

at resonance impedances add up to 0



pure "reactance"

add resistance, damps the pendulum

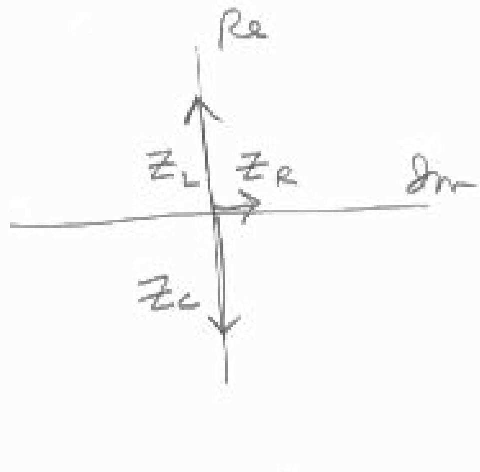


$$Z = j\omega L + \frac{1}{j\omega C} + R =$$

$$\frac{1 - \omega^2 LC}{j\omega C} + R =$$

$$\underbrace{j\left(\frac{\omega^2 LC - 1}{\omega C}\right)}_{\text{pure imaginary}} + R$$

pure Real



can never equal zero