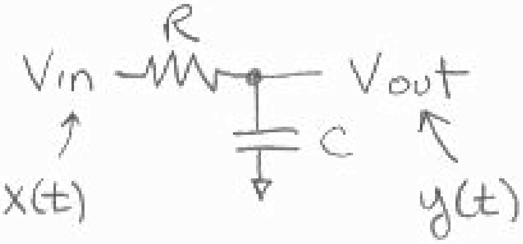
Look at complex impedance ladder as a system



$$Y(\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} X(\omega)$$

$$Y(\omega) = \frac{1}{1 + j\omega RC} X(\omega)$$

Consider each frequency independently:

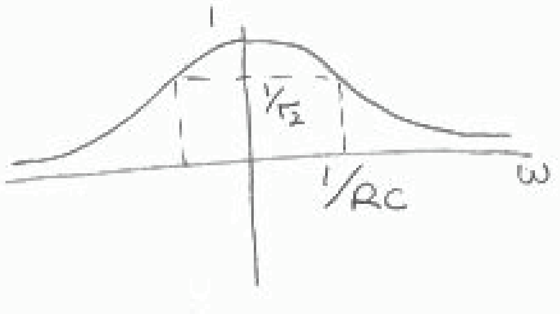
output phasor

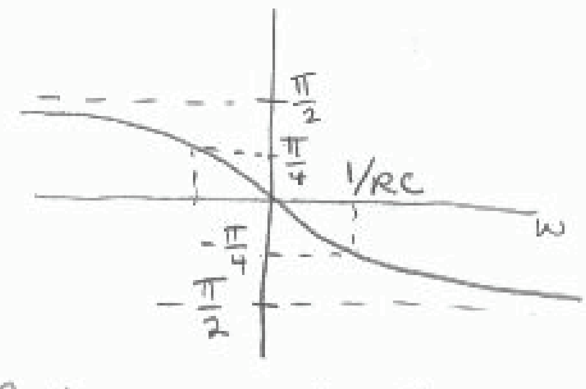
$$Y(\omega)e^{j\omega t} = \frac{1}{1 + j\omega RC} X(\omega)e^{j\omega t}$$

$$Y(\omega) = X(\omega)\,H(\omega), \qquad H(\omega) = \frac{1}{1 + j\omega RC} =$$

$$\frac{1}{1 + j\omega RC} \cdot \frac{1 - j\omega RC}{1 - j\omega RC} = \underbrace{\frac{1}{1 + (\omega RC)^2}}_{Re\{H(\omega)\}} - j\underbrace{\frac{\omega RC}{1 + (\omega RC)^2}}_{Im\{H(\omega)\}}$$

$$|H(\omega)| = \frac{\sqrt{1 + (\omega RC)^2}}{1 + (\omega RC)^2} = \frac{1}{\sqrt{1 + (\omega RC)^2}} \qquad \measuredangle H(\omega) = \tan^{-1}(-\omega RC)$$





at $\omega \ll 1/RC$, capacitor fills up, not there

at $\omega \gg 1/RC$, perfect integrator, 90° phase shift

$|H(\omega)|$ is always even. Why?

Because

$$H(-\omega) = H^*(\omega)$$

at least for real $h(t)$

and

$$|H(\omega)| = |H^*(\omega)|$$

$\measuredangle H(\omega)$ is always odd.

Because

$$\measuredangle H(\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\{H(\omega)\}}{\mathrm{Re}\{H(\omega)\}}\right)$$

since $H(-\omega) = H^*(\omega)$

$$\measuredangle H(-\omega) = \tan^{-1}\left(\frac{-\mathrm{Im}\{H(\omega)\}}{\mathrm{Re}\{H(\omega)\}}\right)$$

since $\tan^{-1}$ is odd

so is $\measuredangle H(\omega)$

Bode Plot (see Hsu p.265)

$$20\log_{10}|H(\omega)| \equiv |H(\omega)|_{dB}$$

$$\sphericalangle H(\omega) \equiv \theta_H(\omega)$$

$\Bigg\}$ vs. ω plotted on log scale

↑ Hsu's NOTATION

20 instead of 10 ("deci") because Bell was measuring power, not amplitude
20dB = 10 times the amplitude ≡ 10^2 times the power

$20\log_{10}|H(\omega)|$ puts $|H(\omega)|$ in <u>deci</u>Bells, dB, which normally represent a ratio, in this case the ratio is:

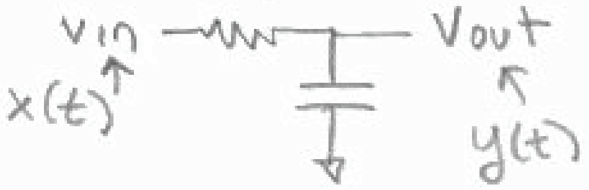
$$|H(\omega)| = \frac{|Y(\omega)|}{|X(\omega)|}$$

While we're at it,

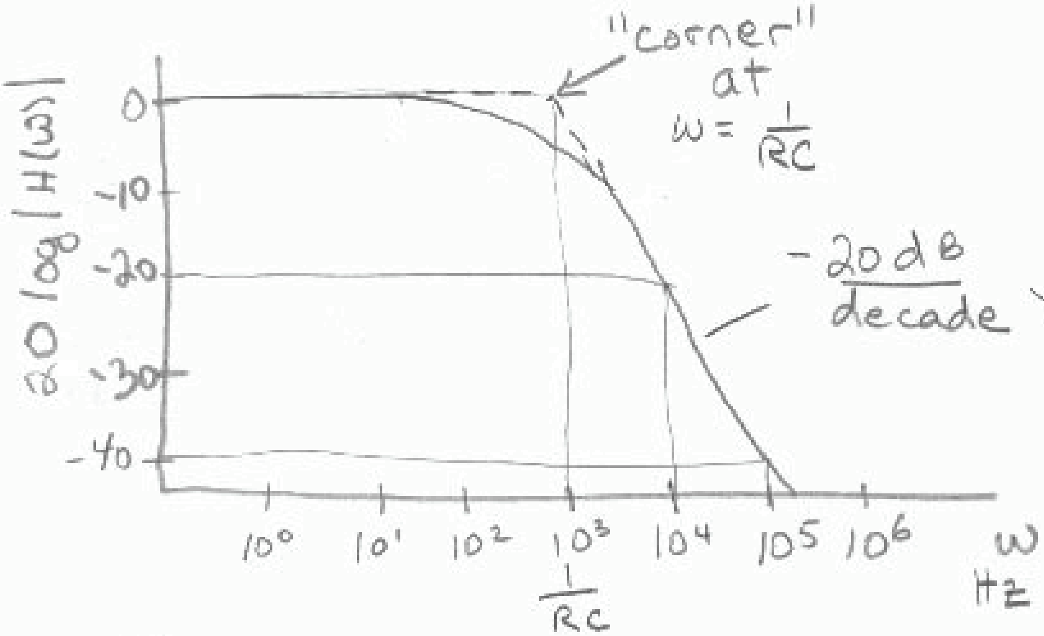
$$\sphericalangle H(\omega) = \sphericalangle Y(\omega) - \sphericalangle X(\omega)$$

Back to <u>Bode</u> <u>Plots</u>

we showed previously



$$H(\omega) = \frac{1}{1 + j\omega RC}$$

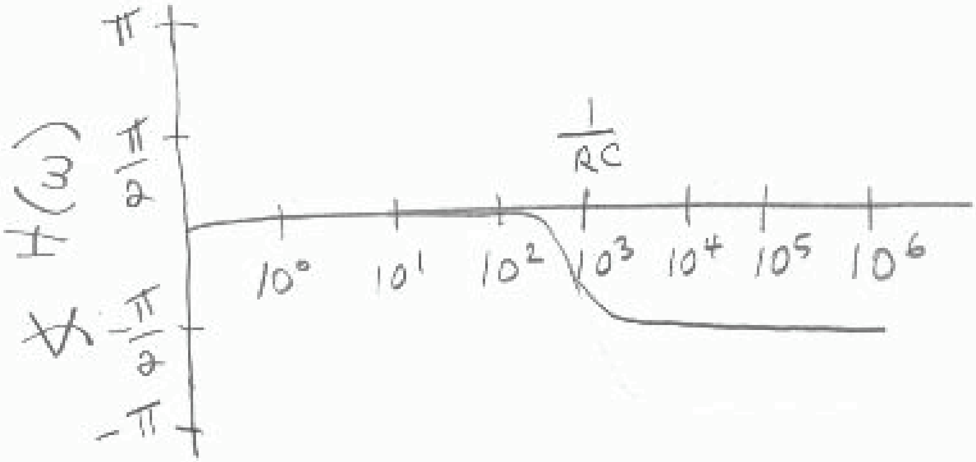


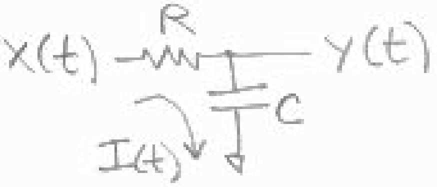
Set
$R = 1\ K\Omega$
$C = 1\ \mu F$
$\frac{1}{RC} = 1\ KHz$

meaning it falls off linearly with frequency.

Sometimes rephrased
$-6 \frac{dB}{octave}$

octave ≡ x2
decade ≡ x10





solving as diff. eq. →

$$I(t) = C\frac{dy(t)}{dt}$$

$$I(t) = \frac{x(t) - y(t)}{R}$$

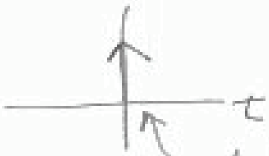
$$y(t) + RC\frac{dy(t)}{dt} = x(t)$$

homogeneous solution with $x(t) = 0$

$$y(t) = e^{-t/RC}$$

↓ let's find the impulse response

$$x(t) = \delta(t)$$



$t = $ "0+" just to right of $t = 0$

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} I(t)\,dt$$

$$I(t) = \frac{\delta(t) - x(t)}{R}$$

Initially, the impulse will charge the capacitor to some finite amount, making the initial $h(t)$

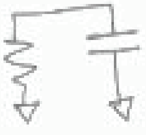
$$h(0+) = \frac{1}{C}\int_{-\infty}^{0+} \frac{\delta(t) - x(t)}{R}\,dt$$

insignicant compared to $\delta(t) = \infty \big|_{t=0}$

$$h(0+) = \frac{1}{C}\int_{-\infty}^{0+} \frac{\delta(t)}{R}\,dt = \frac{1}{RC}$$

Capacitor then discharges through resistor, $x(t) = 0\big|_{t>0+}$ making the whole impulse response

$$h(t) = \frac{1}{RC}e^{-t/RC}u(t)$$



Its Fourier transform is

$$H(\omega) = \int_{-\infty}^{+\infty} \frac{1}{RC}e^{-t/RC}u(t)e^{-j\omega t}\,dt$$

$$= \frac{1}{RC}\int_{0}^{\infty} e^{-\left(\frac{1}{RC}+j\omega\right)t}\,dt$$

$$= -\left[\frac{\frac{1}{RC}}{\frac{1}{RC}+j\omega}\right] e^{-\left(\frac{1}{RC}+j\omega\right)t}\Big|_0^{\infty} = \frac{1}{1+j\omega RC}$$

ta da!