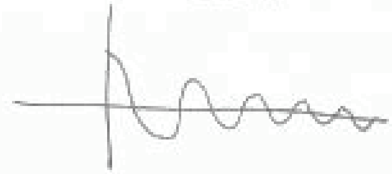
a can be complex

bell

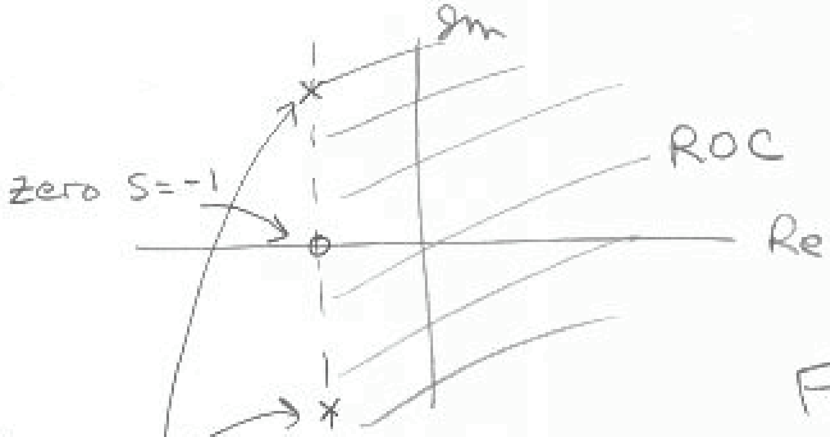
$$x(t) = e^{-t}(\cos 3t)u(t)$$



$$= \left[\frac{1}{2}e^{-\overbrace{(1-3j)}^{a}t} + \frac{1}{2}e^{-(1+3j)t}\right]u(t)$$

$$X(s) = \frac{1}{2}\left[\frac{1}{s+(1-3j)} + \frac{1}{s+(1+3j)}\right]$$

$$= \frac{[s+1]}{[s+(1-3j)]\,[s+(1+3j)]}$$



poles at

$s = -1 \pm 3j$

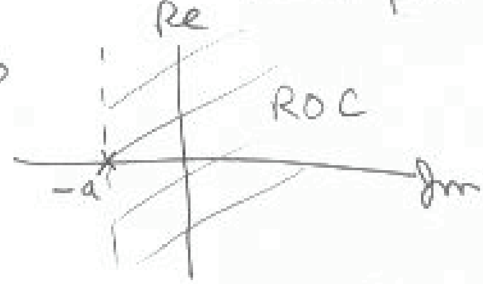
s-plane descibes system stability

Fourier is possible in this example.

Stability of causal system

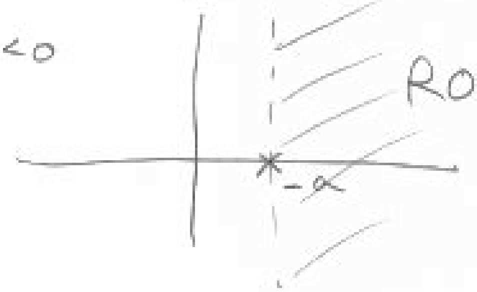
stable only if all poles lie in left half side of s-plane (i.e. all have negative real parts)

$x(t) = e^{-at}u(t) \quad a>0$



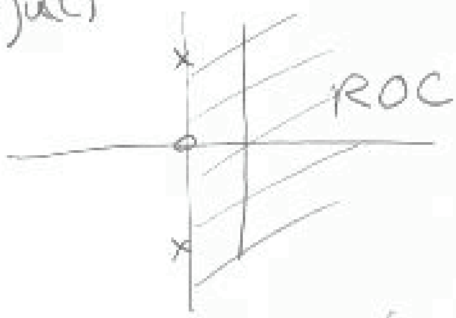
stable

$x(t) = e^{-at}u(t) \quad a<0$



unstable

$x(t) = e^{-t}(\cos 3t)u(t)$

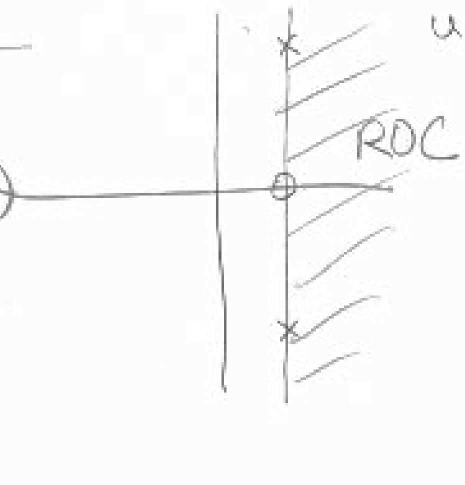


stable

unstabe

$x(t) = e^{t}(\cos 3t)u(t)$

$= \left[\frac{1}{2}e^{-\underbrace{(-1-3j)}_{a}t} + \frac{1}{2}e^{-(-1+3j)t}\right]u(t)$



$X(s) = \frac{1}{2}\left[\frac{1}{s+(-1-3j)} + \frac{1}{s+(-1+3j)}\right]$

$= \frac{[s-1]}{[s+(-1-3j)][s+(-1+3j)]}$

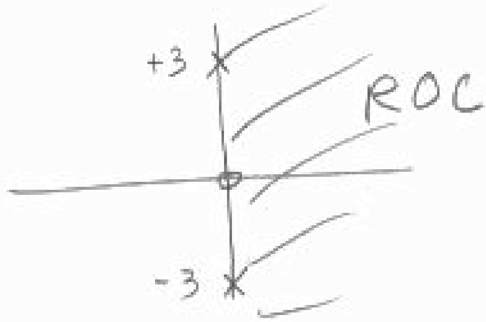
what about

$$x(t) = \cos 3t \, u(t)$$

$$x(t) = \frac{1}{2} e^{-\underbrace{(-3j)}_{a} t} + \frac{1}{2} e^{-(+3j)t}$$

$$X(s) = \frac{1}{2}\left[\frac{1}{s-3j} + \frac{1}{s+3j}\right]$$

$$= \frac{s}{(s-3j)(s+3j)}$$



unstable
bell rings
forever

generally see it written in table this way

$$\cos(\omega_0 t)\, u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+\omega_0^2}$$

$$\sin(\omega_0 t)\, u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0}{s^2+\omega_0^2}$$

properties of $\mathcal{L}$

Linear $\quad A x_1(t) + B x_2(t) \xleftrightarrow{\mathcal{L}} A X_1(s) + B X_2(s)$

Time Shift $\quad x(t-\tau) \xleftrightarrow{\mathcal{L}} e^{-s\tau} X(s)$

Convolution $\quad x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s) X_2(s)$

Differenciation $\quad \frac{d\,x(t)}{dt} \xleftrightarrow{\mathcal{L}} s X(s)$

Integration $\quad \int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s} X(s)$

Scaling $\quad x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{a} X\left(\frac{s}{a}\right) \quad a > 0$ real

---

review analysis

Laplace

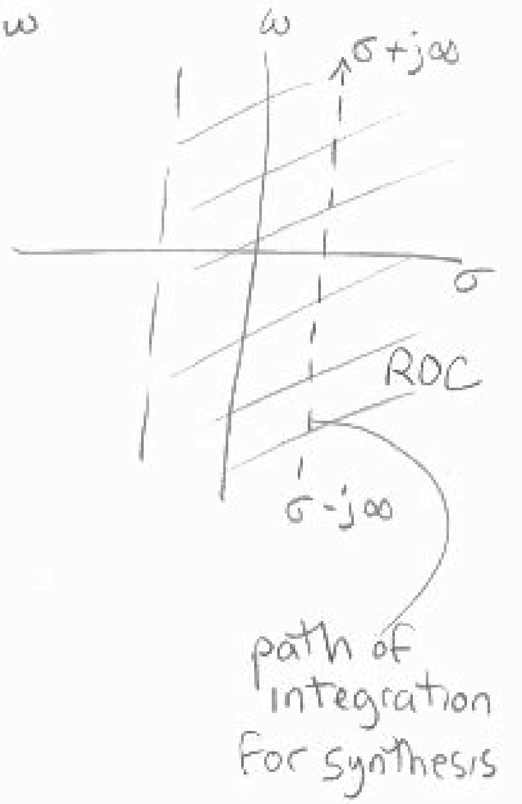
$s = \sigma + j\omega$

$$X(s) = \int_{-\infty}^{+\infty} x(t) e^{-st}\,dt$$

synthesis

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st}\,ds$$

$$X(s) = F\{x(t) e^{-\sigma t}\}$$



Transform Pairs

$$\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1$$

$$\delta(t-\tau) \overset{\mathcal{L}}{\longleftrightarrow} e^{-s\tau}$$

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$$

setting $a=0$

$\Downarrow$

convolution "kernel" for:

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}$$ integration

$$\underbrace{u(t) * u(t) \ldots}_{n \text{ times}} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s^n}$$ multiple integration

$$\frac{d^n\delta(t)}{dt^n} \overset{\mathcal{L}}{\longleftrightarrow} s^n$$

Laplace and Diff eq.

general linear constant coeff. diff. eq., $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

since $\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{L}} sX(s)$, $\Big\updownarrow \mathcal{L}$

$$\left(\sum_{k=0}^{N} a_k s^k\right) Y(s) = \left(\sum_{k=0}^{M} b_k s^k\right) X(s)$$

Since $x(t) * h(t) = y(t) \xleftrightarrow{\mathcal{L}} X(s)\,H(s) = Y(s)$

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

← zeros are solutions to numerator

← poles are solutions to denominator

because for each constituent e^{st}

$$\frac{de^{st}}{dt} = s e^{st}$$

System (example 9.25, oppenheim)

$$x(t) = e^{-3t} u(t)$$

$$y(t) = \left[e^{-t} - e^{-2t} \right] u(t)$$

general model $\quad e^{-at} u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$

$$X(s) = \frac{1}{s+3} \qquad Re\{s\} > -3$$

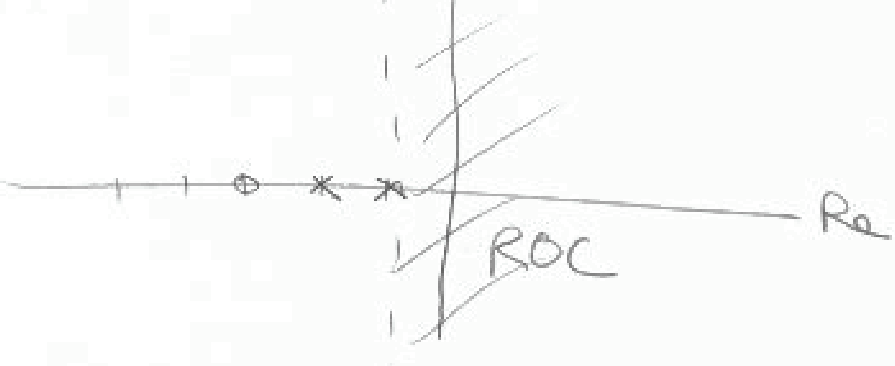
$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$Re\{s\} > -1$$

$$= \frac{(s+2) - (s+1)}{(s+1)(s+2)} = \frac{1}{(s+1)(s+2)}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2} \begin{array}{l} \leftarrow N(s) \\ \leftarrow D(s) \end{array}$$

$$\underbrace{\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t)}_{\text{from } D(s)} = \underbrace{\frac{dx(t)}{dt} + 3x(t)}_{\text{from } N(s)}$$



Partial fractions

$$X(s) = \frac{c_1}{(s+a)} + \frac{c_2}{(s+b)} = \frac{c_1(s+b) + c_2(s+a)}{(s+a)(s+b)}$$

$$\left[X(s)(s+a)\right]_{s=-a} = \left[\frac{c_1(s+b) + c_2(s+a)}{(s+b)}\right]_{s=-a} = c_1$$

How to do inverse transform

From Bruce p. 206

given

$$X(s) = \frac{s+2}{s^2+5s+4}$$

Solve using partial fractions

$$= \frac{s+2}{(s+1)(s+4)} = \frac{C_1}{(s+1)} + \frac{C_2}{(s+4)}$$

"simple" (real) non repeated poles

$$C_1 = \Big[(s+1)X(s)\Big]_{s=-1} = \frac{(s+2)}{(s+4)}\Bigg|_{s=-1} = \frac{1}{3}$$

$$C_2 = \Big[(s+4)X(s)\Big]_{s=-4} = \frac{(s+2)}{(s+1)}\Bigg|_{s=-4} = \frac{2}{3}$$

$$x(t) = \left[\frac{1}{3}e^{-t} + \frac{2}{3}e^{-4t}\right]u(t)$$

3 representations of X(s) above

(a) good for diff eq as we have seen

(b) good for poles + zeros

(c) good for $\longrightarrow x(t)$