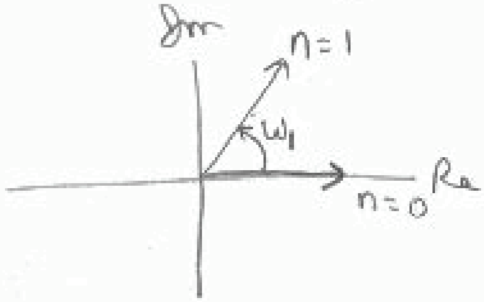
The spectrum of any discrete signal is **Periodic** in **Frequency**!

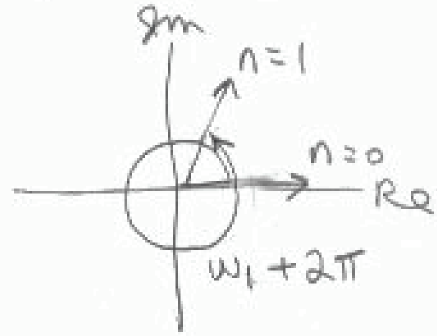
confusing terminology

"Periodic" just means "it repeats", it doesn't necessarily mean "it repeats in time."

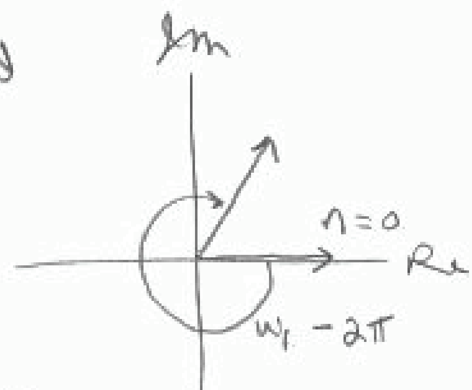
take the discrete phasor, $e^{j\omega_1 n}$, $0 < \omega_1 < \pi$



LOOKS JUST LIKE



LOOKS JUST LIKE



Frequency is no longer Δphase/time, but rather Δphase/sample

in fact

ω_1 is indistinguishable from

$\omega_1 + m2\pi$, $m = 0, \pm 1, \pm 2 \ldots$

$$e^{j\omega_1 n} = e^{j(\omega_1 + m2\pi)n} = e^{j\omega_1 n} \underbrace{e^{jm2\pi n}}_{\substack{\text{m invisible extra revolutions} \\ \text{(forwards or backwards)} \\ \text{per sample}}}$$

same

Discrete Fourier series

<u>synthesis</u>

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \qquad \uparrow \frac{2\pi}{N}$$

period $= N$

sampling frequency $\omega_s = 2\pi$

<u>analysis</u>

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} \qquad \uparrow \frac{2\pi}{N}$$

the k^{th} harmonic is indistinguishable from the $(k+N)^{th}$ harmonic

$$e^{j(k+N)\frac{2\pi}{N}n} = e^{jk\frac{2\pi}{N}n} \underbrace{e^{j2\pi n}}_{1}$$

thus $a_k = a_{k+N}$

periodic in frequency

take the simplest case

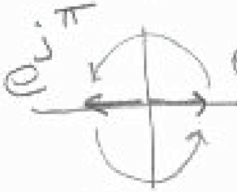
Discrete Fourier Series $N=2$

$x[n]$

root $W = e^{j\frac{2\pi}{N}} = -1$

how far does fundamental spin in 1 sample

$e^{j\pi}$ $e^{j2\pi}$



$a_0 = \frac{1}{2}$

$a_1 = \frac{1}{2}$

Synthesis

$$x[n] = \sum_{\langle N \rangle} a_k e^{jk\omega_0 n}$$

$$\omega_0 = 2\pi/N = \pi$$

$$n = 0, 1$$

1rst harmonic

$$\begin{bmatrix} x[0] \\ x[1] \end{bmatrix} = \begin{bmatrix} W^0 & W^0 \\ W^0 & W^1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

DC

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

FOURIER MATRIX

$$F = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Analysis

$$a_k = \frac{1}{N} \sum_{\langle N \rangle} x[n] e^{-jk\omega_0 n}$$

DC

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (W^*)^0 & (W^*)^0 \\ (W^*)^0 & (W^*)^{-1} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}$$

1rst harmonic

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

INVERSE FOURIER MATRIX

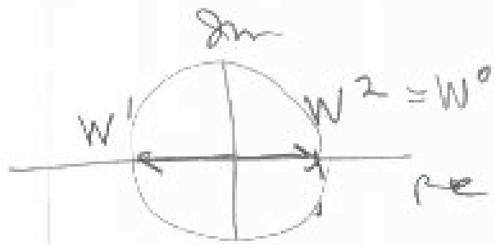
$$F^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$F F^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

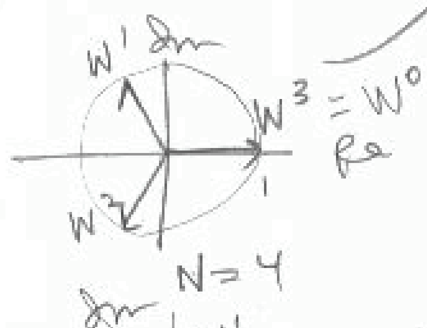
Let W be the N^{th} root of 1

$$W^N = 1 \qquad W = e^{j\frac{2\pi}{N}}$$

eg $N=2$



$N=3$



$N=4$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \end{bmatrix} = \begin{bmatrix} \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \nwarrow & \swarrow \\ \rightarrow & \swarrow & \nwarrow \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$K=2$ harmonic
same as
$K=-1$
harmonic
looks like it's
spinning
backwards

DISCRETE FOURIER SERIES = MATRIX ALGEBRA

(Linear Algebra & its applications, G. Strang p. 183)

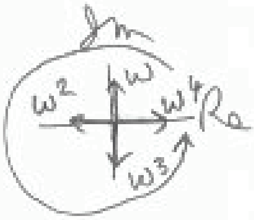
eg. $N = 4$, $W = j$ (4th root of unity)

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & j^2 & j^3 \\ 1 & j^2 & j^4 & j^6 \\ 1 & j^3 & j^6 & j^9 \end{bmatrix}}_{F = \text{"Fourier Matrix"}} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

1'st harmonic

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \underbrace{\frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & (-j) & (-j)^2 & (-j)^3 \\ 1 & (-j)^2 & (-j)^4 & (-j)^6 \\ 1 & (-j)^3 & (-j)^6 & (-j)^9 \end{bmatrix}}_{F^{-1} = \text{"Inverse Fourier matrix"}} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

1'st harmonic



$$FF^{-1} = I$$

$$\begin{bmatrix} \rightarrow & \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \uparrow & \leftarrow & \downarrow \\ \rightarrow & \leftarrow & \rightarrow & \leftarrow \\ \rightarrow & \downarrow & \leftarrow & \uparrow \end{bmatrix} \qquad \frac{1}{4}\begin{bmatrix} \rightarrow & \rightarrow & \rightarrow & \rightarrow \\ \rightarrow & \downarrow & \leftarrow & \uparrow \\ \rightarrow & \leftarrow & \rightarrow & \leftarrow \\ \rightarrow & \uparrow & \leftarrow & \downarrow \end{bmatrix}$$

$O(N^2)$ "order" N^2 calculations

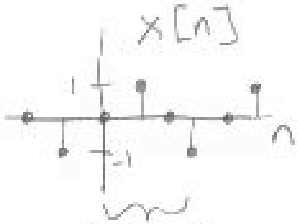
If n is a "magic number" 2^L, can break F down into two matrices of order $\frac{N}{2}$, eg for $N=4$. By successive breakdowns, get $O(N \log N) = O(NL)$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \\ x[1] \\ x[3] \end{bmatrix}$$

<u>Fast</u> <u>Fourier</u> <u>Transform</u>

<u>very</u> important algorithm

need 16, 32, 64 ... 1024 ... samples



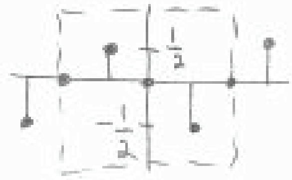
period
N=4

Fs $\longleftrightarrow$

Re{a_k}

baseband

Im{a_k}



baseband

1rst harmonic spinning backwards

$$\begin{bmatrix} a_0=0\\ a_1=-\frac{j}{2}\\ a_2=0\\ a_3=\frac{j}{2}\end{bmatrix} = \frac{1}{4}\underbrace{\begin{bmatrix}1&1&1&1\\1&-j&-1&j\\1&-1&1&-1\\1&j&-1&-j\end{bmatrix}}_{F^{-1}}\begin{bmatrix}x[0]=0\\x[1]=1\\x[2]=0\\x[3]=-1\end{bmatrix}$$

$N=4$
$\omega_0=\frac{\pi}{2}$

also a_{-1}

F^{-1} "Inverse Fourier Matrix" (analysis)

$$a_k=\frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n}$$

$\sin[\omega_0 n]$

1rst harmonic

$$\begin{bmatrix}x[0]=0\\x[1]=1\\x[2]=0\\x[3]=-1\end{bmatrix} = \underbrace{\begin{bmatrix}1&1&1&1\\1&j&-1&-j\\1&-1&1&-1\\1&-j&-1&j\end{bmatrix}}_{F}\begin{bmatrix} a_0=0\\ a_1=-\frac{j}{2}\\ a_2=0\\ a_3=\frac{j}{2}\end{bmatrix}$$

F "Fourier Matrix" (Synthesis)

$$x[n]=\sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$