$$X_s(\omega) = X_s(\omega + k\omega_s) \qquad k = 0, \pm1, \pm2, \ldots$$

spectrum of sampled signal

adds k hidden extra complete revolutions

periodic in frequency ω

period = ω_s

we need only one period, the "baseband"

$-\frac{\omega_s}{2} < \omega < \frac{\omega_s}{2}$ to recreate $x(t)$

if $|\omega_m| < \frac{\omega_s}{2}$ $x(t)$ will be unaltered

Fourier synthesis

$$x[n] = \frac{1}{2\pi}\int_{-\frac{\omega_s}{2}}^{+\frac{\omega_s}{2}} X(\omega)\, e^{j\omega n}\, d\omega$$

BioE 1410 - Sampling and Aliasing handout **Stetten**

Consider the Fourier Transform $X(\omega)$ of a signal $x(t)$ sampled at frequency ω_s. Comparing $X(\omega)$ to $X(\omega+\omega_s)$, we find that adding ω_s adds a complete extra revolution to each phasor as you step from one sample of $x(t)$ and the next. After all, ω_s is by definition one revolution per sample. Performing a complete extra revolution on a phasor leaves its value unchanged. Therefore,

$$X(\omega) = X(\omega + \omega_s) \tag{1}$$

In fact, performing any integer $\pm k$ number of extra revolutions on any of the phasors that constitute $x(t)$ leaves it unchanged, so

$$X(\omega) = X(\omega + k\omega_s), \quad k = 0, \pm 1, 2, ... \tag{2}$$

Therefore, $X(\omega)$ is periodic in the frequency domain, with the period being equal to ω_s. It may seem strange that a "period" can be a frequency, but remember that $X(\omega)$ is a function of frequency.

To recreate $x(t)$ we only need one period of $X(\omega)$, which we can isolate using a band-pass filter between $-\frac{\omega_s}{2}$ and $\frac{\omega_s}{2}$. We shall call this period the *base-band*, because it is that frequency band which includes the frequency $\omega = 0$. The inverse Fourier transform for $X(\omega)$ is found by integrating over that (or any other) single period of $X(\omega)$,

$$x(t) = \int_{-\frac{\omega_s}{2}}^{+\frac{\omega_s}{2}} X(\omega) e^{j\omega t} d\omega . \tag{3}$$

If the original signal $x(t)$ contains only frequencies inside the base-band before it is sampled, those frequencies will show up unaltered in the base-band of $X(\omega)$ after sampling. Copies will appear in neighboring periods at $\pm k\omega_s$. For example, in

$$x(t) = \sin\left(\frac{1}{4}\omega_s t\right) \tag{4}$$

(see Fig. 1) the original phasors (in bold) fall in the base-band of $X(\omega)$ between $-\frac{\omega_s}{2}$ and $\frac{\omega_s}{2}$ (dotted box). Copies of the phasors appear in neighboring periods.

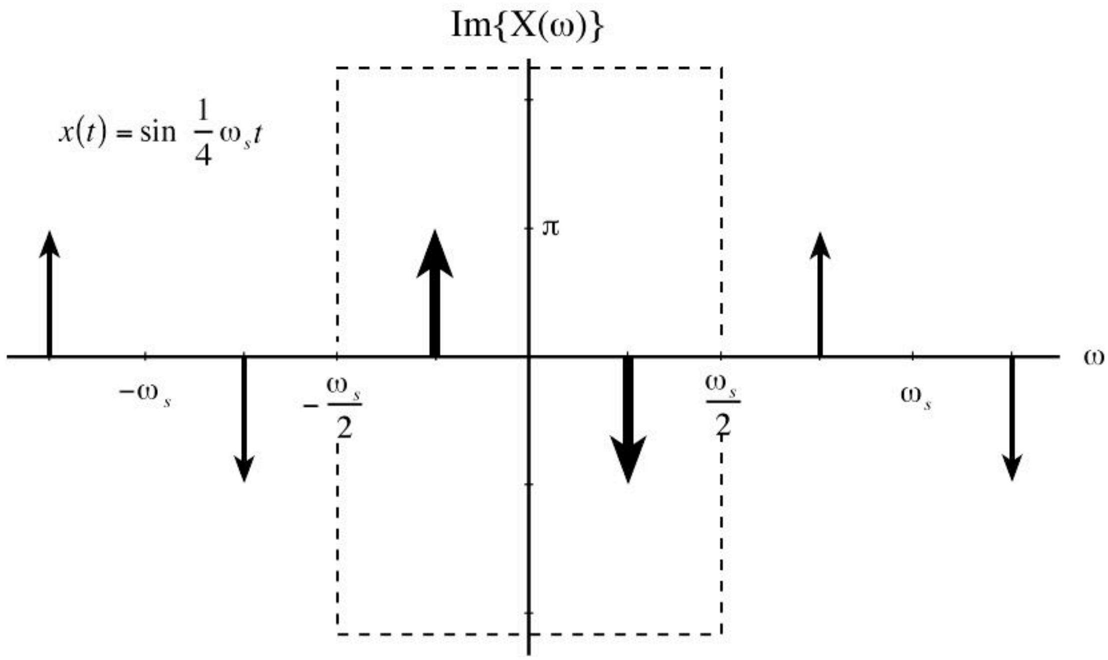


Fig. 1 No aliasing occurs when original phasors (bold arrows) occur at less than half the sampling frequency, falling within the base-band (dotted box) of $\mathrm{Im}\{X(\omega)\}$.

However, if phasors occur at frequencies $|\omega| > \frac{\omega_s}{2}$ outside the base-band, neighboring copies will extend back into the base period and *aliasing* will occur. Such is the case in Fig. 2 with

$$x(t) = \sin\left(\frac{3}{4}\omega_s t\right). \tag{5}$$

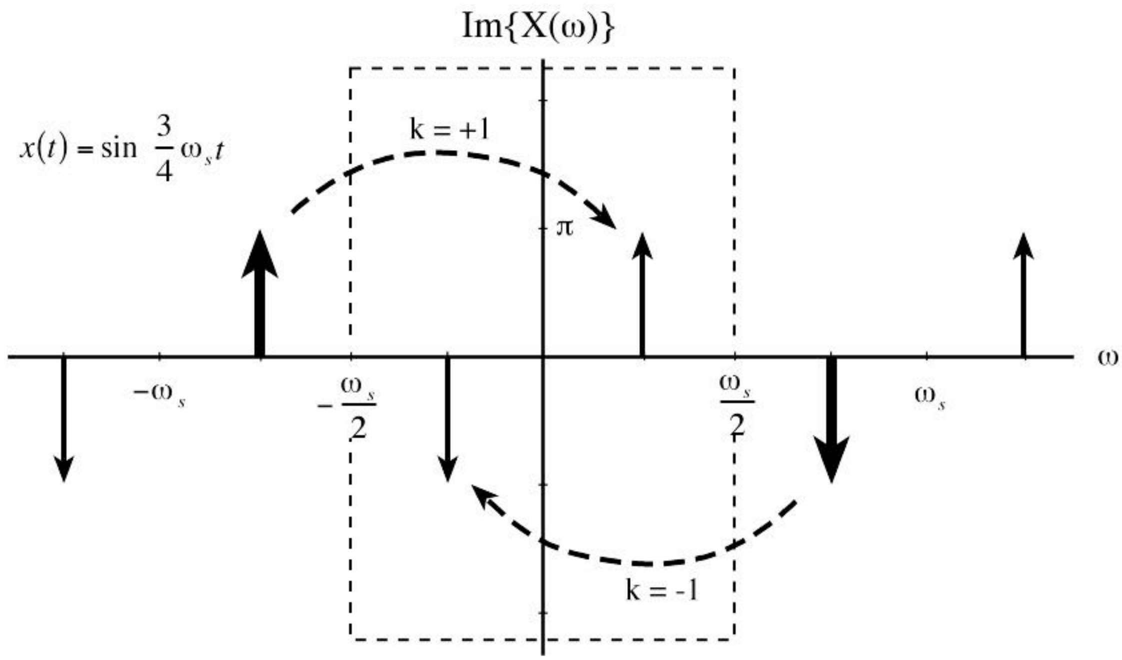


Fig. 2 Aliasing occurs when original frequencies (bold arrows) are greater than half the sampling frequency, falling outside the base period (dotted box) of $\mathrm{Im}\{X(\omega)\}$.

The original phasors (bold) fall outside the base period, and copies from neighboring periods of $X(\omega)$ show up in the base period, creating an *alias* signal that by inspection is

$$-\sin\left(\frac{1}{4}\omega_s t\right) \tag{6}$$

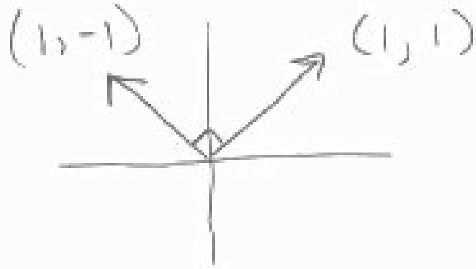
Note the negative sign, as see earlier in the lecture when plotting the sampled signal in the time domain.

k=0, DC

k=1, fundamental

$$F = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

orthogonal vectors



$\overbrace{\text{dot product}}^{\text{orthogonal}} = 0$, acounts for 0's in Identity matrix.

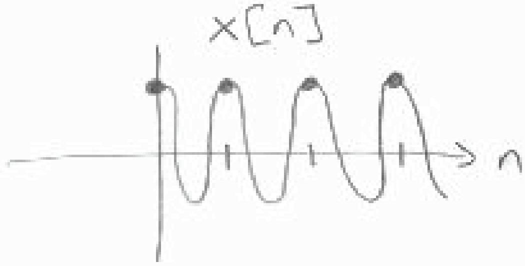
example 1 $\quad x[n] = 6$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 6 \end{bmatrix} \begin{matrix} x[0] \\ x[1] \end{matrix}$$

$a_0 = 6$

$a_1 = 0$

example 2 $\quad x[n] = \cos[2\pi n]$

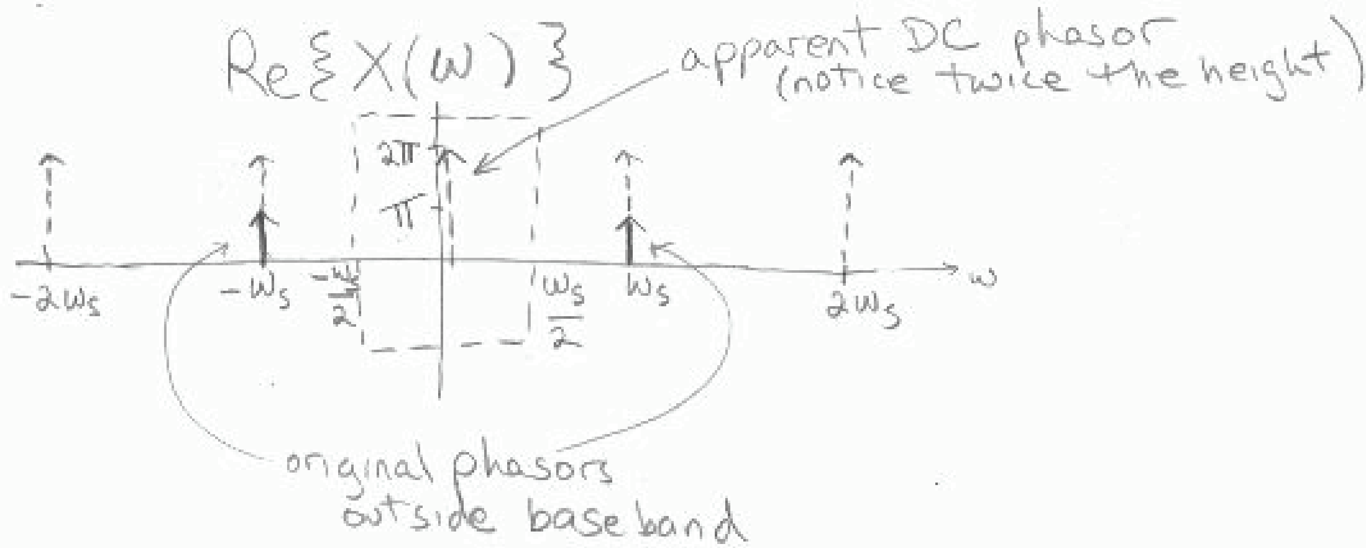


$\omega_M = 2\pi$

$\omega_s = 2\pi$

$|\omega_M| \not< \frac{\omega_s}{2}$

Niquist criterion not satisfied



example 3 $\quad x[n] = \cos[\pi n]$

$$\omega_M = \frac{\omega_s}{2} \qquad \omega_s = 2\pi$$

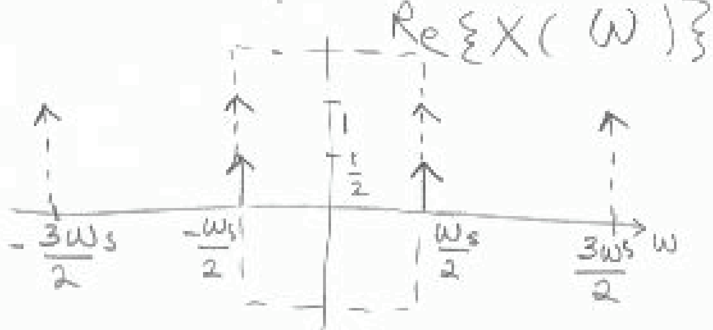
NIQUIST CRITERION NOT SATISFIED

$$\boxed{|W_m| < \frac{W_s}{2}}$$

REMEMBER NOT "≤"

BUT IT SEEMS TO WORK!

BUT KEEP READING...

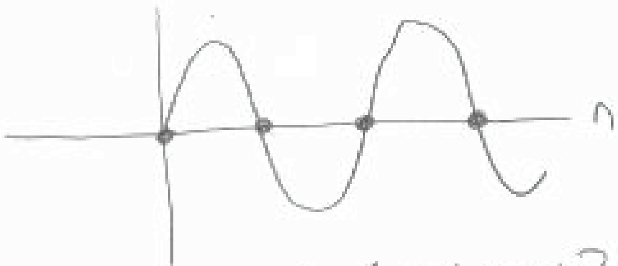


example 4 $\quad x[n] = \sin[\pi n]$

$$\omega_m = \frac{\omega_s}{2} \qquad \omega_s = 2\pi$$

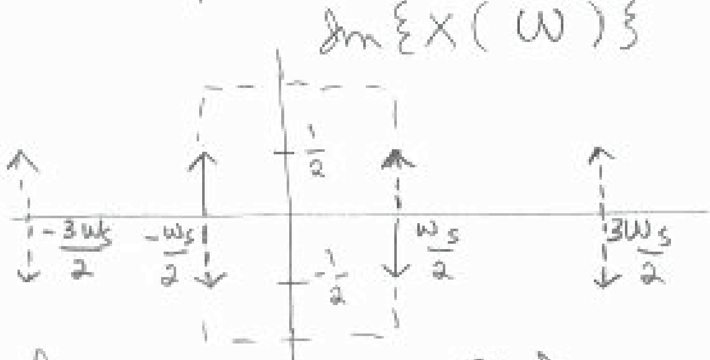
SAME FREQ AS EXAMPLE 3 BUT DIFFERENT PHASE

LOOK OUT!



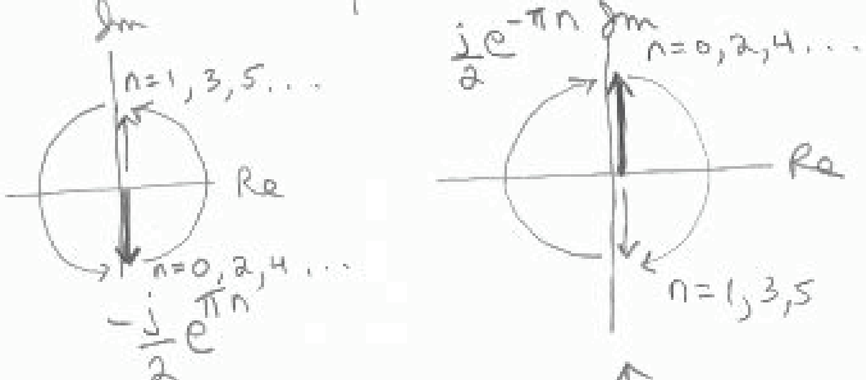
the signal has disappeared!

What happened?



The solution is "quadrature detection".

Look for sin and cos seperately.



forward and backward spinning phasors are 180° apart at sample time

null points in destructive interference pattern

Z-Transform

(Laplace Transform of discrete signals)

in continuous Laplace, we stabilized $x(t)$ by multiplying by $e^{-\sigma t}$

$$X(s) = X(\sigma + j\omega) = \underbrace{\int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}] e^{-j\omega t} dt}_{F\{x(t)e^{-\sigma t}\}} = \int_{-\infty}^{+\infty} x(t) e^{-st} dt =$$

in discrete Laplace (z-transform) we stabilize $x[n]$ by multiplying by r^{-n}, where r is real and positive.

$x[n]r^{-n}$ what is its Fourier Transform?

The Fourier Transform becomes

$$\mathcal{F}\{x[n]r^{-n}\} = \sum_{n=-\infty}^{+\infty} \{x[n]r^{-n}\} e^{-j\omega n} =$$

$$\sum_{n=-\infty}^{+\infty} x[n] (\underbrace{re^{j\omega}}_{z})^{-n} = X(z)$$

Z - transform

discrete time counterpart of Laplace Xform

$$x[n] \overset{Z}{\longleftrightarrow} X(z)$$

$$\boxed{X(z) \triangleq \sum_{n=-\infty}^{+\infty} x[n] z^{-n}}$$

constant

$$z = re^{j\omega}$$

"z-plane"

$z = e^{j\omega}$ on unit circle, when $r = 1$

reduces to discrete Fourier Transform

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

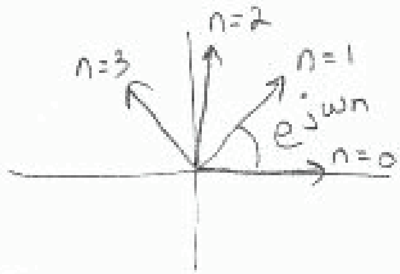
Inverse Z-transform (synthesis)

starting with inverse discrete Fourier Transform

synthesis

$$x[n] = \frac{1}{2\pi}\int_{2\pi} \underbrace{X(\omega)}_{\text{continuous but periodic in frequency domain}} \underbrace{e^{j\omega n}} \, d\omega$$

as n progresses $e^{j\omega}$ is raised to successive integer powers



for a given ω

phase sample

We showed how

$$x[n]r^{-n} \overset{F}{\longleftrightarrow} X(z), \quad z = re^{jw}$$

attenuation by backward spinning phasor in z analysis

$$x[n] = r^n \underbrace{\frac{1}{2\pi}\int_{2\pi} X(z)\, e^{jwn}\, dw}_{\text{Fourier synthesis}}$$

r^n is constant, can move inside the integration.

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(z) \underbrace{(re^{jw})^n}\, dw$$

for a given n, this which is $z^n = r^n e^{jwn}$, traces n circles of radius r^n in the z-plane as a function of $-\pi < w < \pi$.

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(z)\, z^n\, dw$$

at step n this is a rotation $n \times w$ and scale r^n

2π ← z^n goes n times around the circle

with r fixed, $\frac{dz}{dw} = jre^{jw}$, $dz = jz\,dw$

$$dw = \frac{dz}{jz}$$

$$\boxed{x[n] = \frac{1}{2\pi j}\oint X(z)\, z^{n-1}\, dz}$$

Inverse Z transform

Review of geometric series

<u>infinite</u>

for $|\alpha| < 1$

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$$

<u>finite</u>

for any α

$$\sum_{n=0}^{N} \alpha^n = \frac{1-\alpha^{N+1}}{1-\alpha}$$

example

$$x[n] = a^n u[n]$$

$$\boxed{X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n}}$$

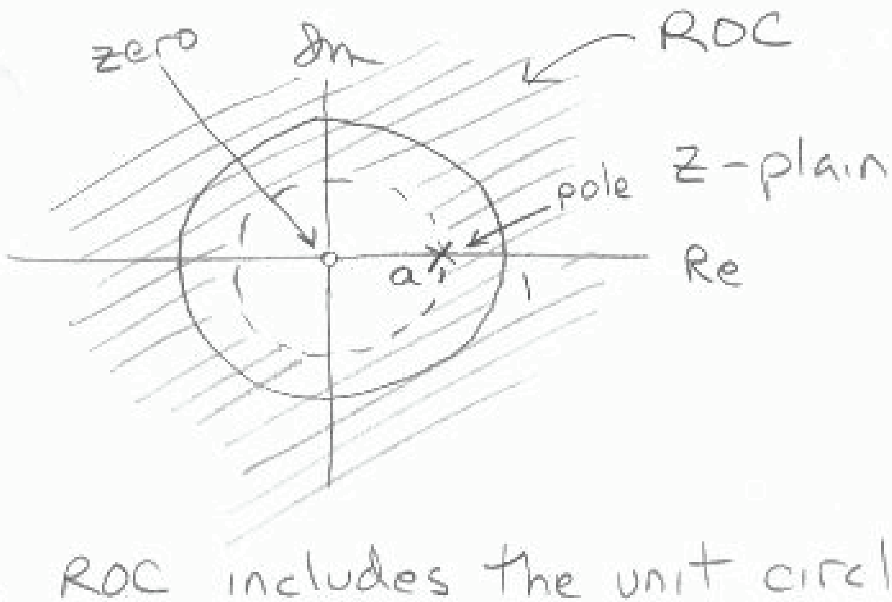
converges for $|z| > |a|$

$$= \sum_{n=0}^{+\infty} a^n z^{-n} = \sum_{n=0}^{+\infty} \left(\frac{a}{z}\right)^n$$

$$= \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}$$

$$a^n u[n] \xleftrightarrow{z} \frac{z}{z-a}$$

for $0 < a < 1$



if $x[n]$ is right-sided ROC lies outside the pole most distant from origin

Since ROC includes the unit circle, Fourier works, $(r=1)$

when $r=1$

$$X(\omega) = \sum_{-\infty}^{+\infty} x[n] e^{-j\omega n}$$

example

$$x[n] = \delta[n]$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \delta[n] z^{-n} = 1$$

piece of wire

ROC = whole z-plane

no zeros no poles

$$x[n] * y[n] \overset{z}{\longleftrightarrow} X(z)Y(z)$$

$$x[n-n_0] \overset{z}{\longleftrightarrow} z^{-n_0} X(z)$$

example

$$x[n] = \delta[n-m]$$

$$X(z) = \sum_{n=-\infty}^{+\infty} \delta[n-m] z^{-n} = z^{-m}$$

eg when $m = 1$

$$\delta[n-1] \longleftrightarrow z^{-1}$$

So, since Z transform is linear, the difference operator

$$\delta[n] - \delta[n-1] = 1 - z^{-1}$$

(compare with $x[n] = u[n]$)

since $a^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{z-a} = \frac{1}{1-az^{-1}}$

when $a = 1$

$$u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-z^{-1}}$$

inverse

If this is the impulse response

you have "Summation" - discrete version of integration

$$x[n] \longrightarrow \boxed{\delta[n]-\delta[n-1]} \longrightarrow \boxed{u[n]} \longrightarrow x[n]$$

$$\underbrace{x[n] \longrightarrow \boxed{1-z^{-1}} \longrightarrow \boxed{\frac{1}{1-z^{-1}}}}_{H(z) = 1 \quad h[n] = \delta[n]} \longrightarrow x[n]$$

two examples

$$x[n] = u[n]$$

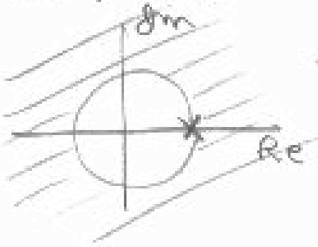
$$X(z) = \sum_{n=-\infty}^{+\infty} u[n] z^{-n} = \sum_{n=0}^{+\infty} z^{-n} = \sum_{n=0}^{+\infty} (z^{-1})^n$$

$$X(z) = \frac{1}{1-\alpha}$$

$$X(z) = \frac{1}{1-z^{-1}}$$

by infinite geometric series $\leftarrow$ call z^{-1} "α"

z-plane



pole at $z=1$

system is an integrator so unstable at D.C.

$$x[n] = u[n] - u[n-3]$$

$$X(z) = \sum_{n=0}^{2} z^{-n}$$, again, call z^{-1} "α"

$$X(z) = \frac{1-\alpha^{N+1}}{1-\alpha} = \frac{1-z^{-3}}{1-z^{-1}}$$

by finite geometric series

pole and zero at $z=1$ cancel.

z-plane

ROC is whole z plane

system is clearly stable FIR,

Averages last 3 samples